Last Time

$$-\Box \varphi = \rho$$

$$-\Box \vec{A} = \vec{J}/c$$

Then knowing the Grn-fcn of the wave-eqn.

exact
$$\begin{cases} \varphi = \int_{r_0} \dfrac{\rho(T, r_0)}{4\pi|\vec{r}-\vec{r}_0|} \\ \vec{A} = \int_{r} \dfrac{\vec{J}(T, r_0)/c}{4\pi|\vec{r}-\vec{r}_0|} \end{cases}$$

$$T = t - \frac{|\vec{r}-\vec{r}_0|}{c}$$

Last time we considered the exact fields of a magnetic dipole. And we saw the transition from near field (see Sec

$$B(t,r) = \left.\frac{3\vec{n}(\vec{n}\cdot\vec{m}) - \vec{m}}{4\pi r^3}\right|_{T} \quad \text{(near field)}$$

$T \leftarrow$ time

To far field

$$B(t,r) = \left.\frac{-1}{4\pi r c^2}(\ddot{\vec{m}} - \vec{n}(n\cdot\ddot{\vec{m}}))\right|_{T} \quad \text{(far field)}$$

powers of $\frac{1}{r}$ of near field (quasi-static)

result replaced with $\frac{1}{c}\frac{\partial}{\partial t}$

If all we care about is the far field we have $T \simeq t - r/c + n \cdot r_o/c$

$$\varphi = \frac{1}{4\pi r} \int_{r_o} \rho\left(t - \frac{r}{c} + \frac{n \cdot r_o}{c}, r_o\right)$$

$$\vec{A} = \frac{1}{4\pi r} \int_{r_o} \frac{\vec{J}}{c}\left(t - \frac{r}{c} + \frac{n \cdot \vec{r}_o}{c}, \vec{r}_o\right) \Leftarrow \text{All you need}$$

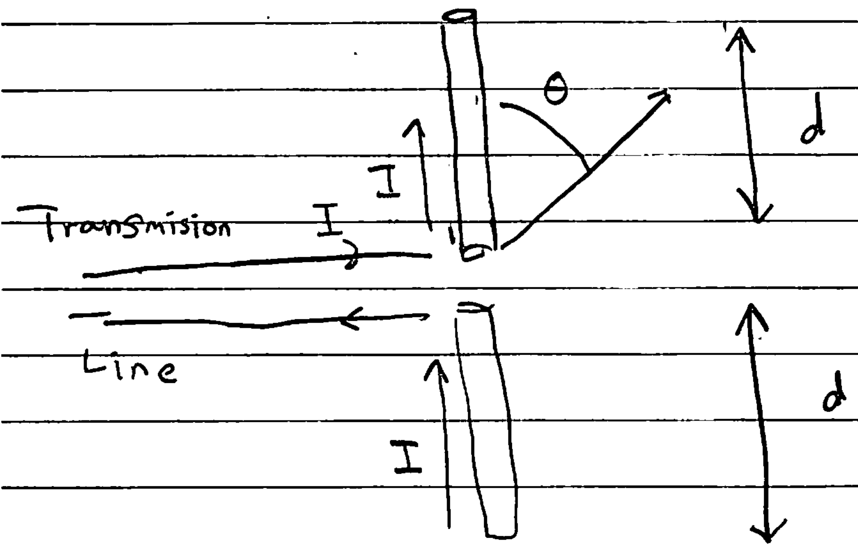
Then we expanded J and interpreted the result

$$J\left(t - \frac{r}{c} + \frac{n \cdot \vec{r}_o}{c}\right) \simeq \underbrace{J\left(t - \frac{r}{c}\right)}_{\text{e-dipole}} + \underbrace{\frac{n \cdot r_o}{c} \dot{J}\left(t - \frac{r}{c}\right)}_{\text{m-dipole + e-quad.}}$$

<u>Today</u>

- Radio - Antennas
- Frequency spectrum for general currents

Linear Antennas (Center-Fed)



- The current goes to zero at the end
- Take a sinusoidal current

$$\vec{J}(t_0, \vec{r}_0) = \underbrace{I_0 \sin(\overset{k=\omega/c}{k}d - k|z_0|)}_{\text{Standing wave of current}} \hat{z}\, \delta(x_0)\, \delta(y_0)\, e^{-i\omega t_0}$$

Then

$$A(t,r) = \frac{1}{4\pi r}\int_{r_0} J(\underset{T = t - \frac{r}{c} + \frac{n \cdot r_0}{c}}{T}, r_0)/c$$

$$= I_0 \frac{e^{-i\omega(t - r/c)}}{4\pi r} \int_{-d}^{d} e^{-\frac{i\omega n \cdot \vec{r}_0}{c}} \sin(kd - k|z_0|)\, \hat{z}\, dz_0$$

Using $\frac{-i\omega}{c} n \cdot r_0 = -ikz_0 \cos\theta$

$$\vec{A}(t,r) = \frac{I_0}{c} \frac{e^{-i\omega(t-r/c)}}{4\pi r} \hat{z} \int_{-d}^{d} e^{-ikz_0\cos\theta} \sin(kd - k|z_0|)\, dz_0$$

$$= \frac{I_0}{c} \frac{e^{-i\omega(t-r/c)}}{4\pi r} \hat{z} \left[\frac{2}{k} \left(\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin^2\theta} \right) \right]$$

So the power radiated

$$\frac{dP}{d\Omega} = c\,|r\vec{E}|^2$$

$$= c \left[r \left(\frac{-1}{c} \frac{\partial \vec{A}_T}{\partial t} \right) \right]^2 \qquad A_T = \vec{A} - \vec{n}(n\cdot\vec{A})$$

$$\frac{-1}{c}\frac{\partial A_T}{\partial t} = ik A_T$$

$$\left(\frac{dP}{d\Omega}\right)_{\text{time ave}} = \frac{c}{2} \left[r^2 k^2 A_T(k) A_T^*(k) \right]$$

$$\boxed{\overline{\frac{dP}{d\Omega}} = \frac{c}{8\pi^2} \left(\frac{I_0}{c}\right)^2 \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2}$$

Comments

① Previously we derived a multipole expansion valid when $kd \ll 1$, i.e. when

$$\frac{2\pi d}{\lambda} \ll 1$$

Thus when $kd \ll 1$ should recover the dipole limit. Indeed expanding for $kd \ll 1$

$$\overline{\frac{dP}{d\Omega}} = \frac{c}{16\pi^2} \underbrace{\left(\frac{I_0}{c}\right)^2 (kd)^4 \sin^2\theta}$$

See a characteristic dipole field and frequency dependence

② The total power can be determined

$$\bar{P} = \frac{1}{2} c \left(\frac{I_0}{c}\right)^2 \int \frac{d\Omega}{4\pi^2} \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta}\right]^2$$

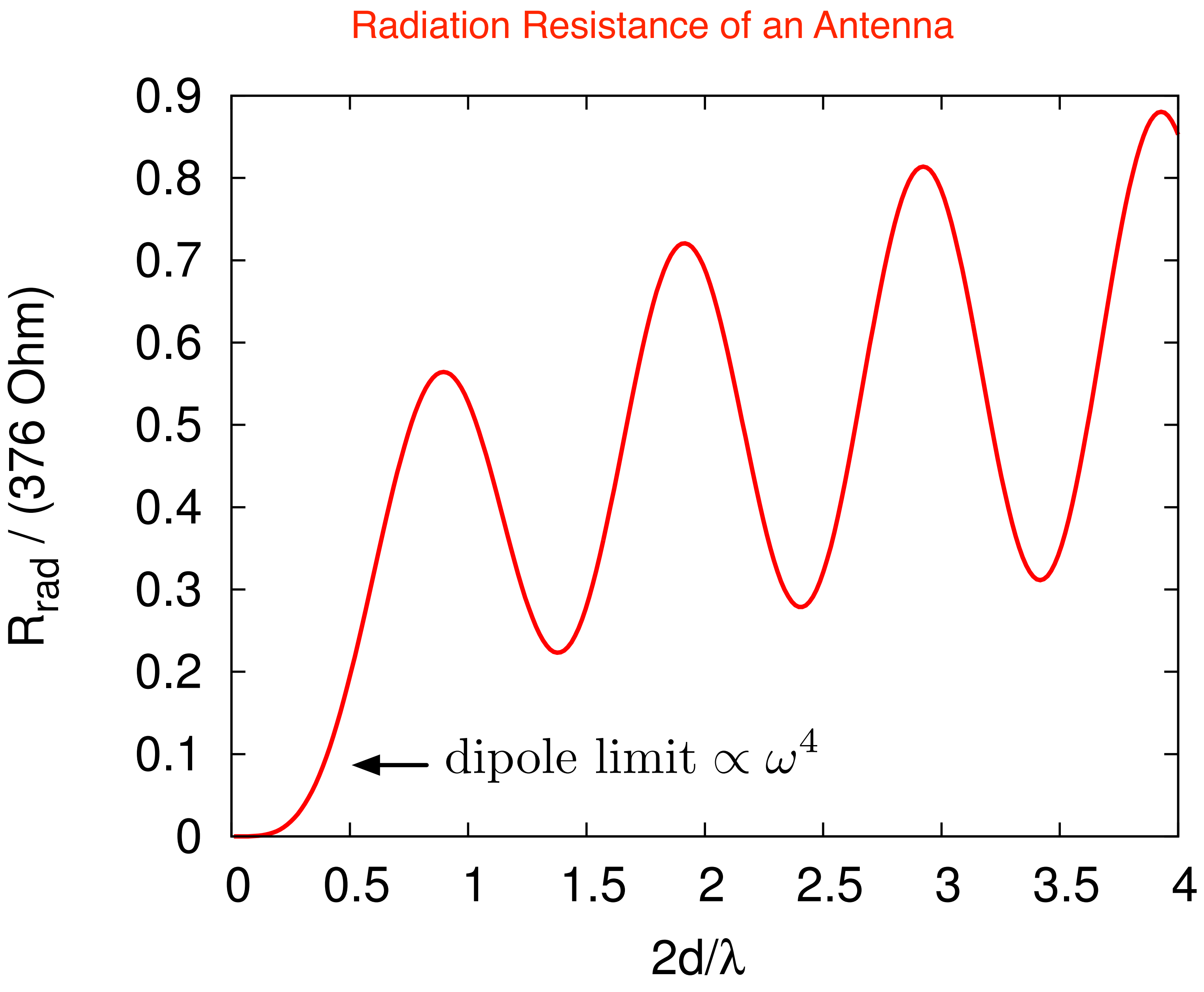
mks

$$\bar{P} = \frac{1}{2} \underbrace{c\mu_0} I_{mks}^2 \, f(kd)$$

have to do numerically

this factor $= \sqrt{\frac{\mu_0}{\varepsilon_0}}$ comes up all the time

and is the "Impedance of Vacuum = 376 Ω"


Radiation Resistance of an Antenna
R_{rad} / (376 Ohm)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0
0.5
1
1.5
2
2.5
3
3.5
4
2d/λ
dipole limit $\propto \omega^4$

Then can write.

$$P = \frac{1}{2} R_{rad} \, I^2_{mks}$$

Where the radiation resistance is

$$R_{rad} = 376\,\Omega \int \frac{d\Omega}{4\pi^2} \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2$$

Find numerically that