11 Radiation in Non-relativistic Systems

11.1 Basic equations

This first section will *NOT* make a non-relativistic approximation, but will examine the far field limit.

(a) We wrote down the wave equations in the covariant gauge:

$$-\Box\Phi = \rho(t_o, \boldsymbol{r}_o) \tag{11.1}$$

$$-\Box\boldsymbol{A} = \boldsymbol{J}(t_o, \boldsymbol{r}_o)/c \tag{11.2}$$

The gauge condition reads

$$\frac{1}{c}\partial_t\Phi + \nabla\cdot\boldsymbol{A} = 0 \tag{11.3}$$

(b) Then we used the green function of the wave equation

$$G(t, r|t_o r_o) = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}\delta(t - t_o + \frac{|\boldsymbol{r}-\boldsymbol{r}_o|}{c}) \tag{11.4}$$

to determine the potentials $(\Phi, \boldsymbol{A})$

$$\Phi(t, \boldsymbol{r}) = \int \mathrm{d}^3x_o \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}\rho(T, \boldsymbol{r}_o) \tag{11.5}$$

$$\boldsymbol{A}(t, \boldsymbol{r}) = \int \mathrm{d}^3x_o \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}\boldsymbol{J}(T, \boldsymbol{r}_o)/c \tag{11.6}$$

Here $T(t, \boldsymbol{r})$ is the retarded time

$$T(t, r) = t - \frac{|\boldsymbol{r}-\boldsymbol{r}_o|}{c} \tag{11.7}$$

(c) We used the potentials to determine the electric and magnetic fields. Electric and magnetic fields in the far field are

$$\boldsymbol{A}_{\mathrm{rad}}(t, \boldsymbol{r}) = \frac{1}{4\pi r}\int_{\boldsymbol{r}_o} \frac{\boldsymbol{J}(T, \boldsymbol{r}_o)}{c} \tag{11.8}$$

and

$$\boldsymbol{B}(t, \boldsymbol{r}) = -\frac{\boldsymbol{n}}{c}\times\partial_t\boldsymbol{A}_{\mathrm{rad}} \tag{11.9}$$

$$\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{n}\times\frac{\boldsymbol{n}}{c}\times\partial_t\boldsymbol{A}_{\mathrm{rad}} = -\boldsymbol{n}\times\boldsymbol{B}(t, \boldsymbol{r}) \tag{11.10}$$

In the far field (large distance limit $\boldsymbol{r}\to\infty$) limit we have

$$T = t - \frac{r}{c} + \boldsymbol{n}\cdot\frac{\boldsymbol{r}_o}{c} \tag{11.11}$$

And we recording the derivatives

$$\left(\frac{\partial}{\partial t}\right)_{r_o} = \left(\frac{\partial}{\partial T}\right)_{r_o} \tag{11.12}$$

$$\left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_t = \left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_T + \frac{\boldsymbol{n}}{c}\left(\frac{\partial}{\partial T}\right)_{\boldsymbol{r}_o} \tag{11.13}$$

(d) We see that the radiation (electric field) is proportional to the transverse piece of the $\partial_t \boldsymbol{J}$

$$-\boldsymbol{n} \times (\boldsymbol{n} \times \partial_t \boldsymbol{J}) = \partial_t \boldsymbol{J} - \boldsymbol{n}(\boldsymbol{n} \cdot \partial_t \boldsymbol{J}) \tag{11.14}$$

In general the transverse projection of a vector is

$$-\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{V}) = \boldsymbol{V} - \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{V}) \tag{11.15}$$

(e) Power radiated per solid angle is for $r \to \infty$ is

$$\frac{dW}{dtd\Omega} = \frac{dP(t)}{d\Omega} = \text{energy per observation time per solid angle} \tag{11.16}$$

and

$$\frac{dP(t)}{d\Omega} = r^2 \boldsymbol{S} \cdot n \tag{11.17}$$

$$= c|rE|^2 \tag{11.18}$$

11.2 Examples of Non-relativistic Radiation: L31

In this section we will derive several examples of radiation in non-relativistic systems. In a non-relativistic approximation

$$T = t - \frac{r}{c} + \underbrace{\frac{\boldsymbol{n}}{c} \cdot \boldsymbol{r}_o}_{\text{small}} \tag{11.19}$$

The underlined terms are small: If the typical time and size scales of the source are T_{typ} and L_{typ}, then $t \sim T_{\text{typ}}$, and $\boldsymbol{r}_o \sim L_{\text{typ}}$, and the ratio the underlined term to the leading term is:

$$\frac{L_{\text{typ}}}{cT_{\text{typ}}} \ll 1 \tag{11.20}$$

This is the non-relativistic approximation. For a harmonic time dependence, $1/T_{\text{typ}} \sim \omega_{\text{typ}}$, and this says that the wave number $k = \frac{2\pi}{\lambda}$ is small compared to the size of the source, *i.e. the wave length of the emitted light is long compared to the size of the system in non-relativistic motion:*

$$\frac{2\pi L_{\text{typ}}}{\lambda} \ll 1 \tag{11.21}$$

(a) Keeping only $t - r/c$ and dropping all powers of $\boldsymbol{n} \cdot \boldsymbol{r}_o / c$ in T results in the electric dipole approximation, and also the Larmour formula.

(b) Keeping the first order terms in

$$\frac{\boldsymbol{n}}{c} \cdot \boldsymbol{r}_o \tag{11.22}$$

results in the magnetic dipole and quadrupole approximations.

The Larmour Formula

(a) For a particle moves slowly with velocity and acceleration, $\boldsymbol{v}(t)$ and $\boldsymbol{a}(t)$ along a trajectory $\boldsymbol{r}_*(t)$

(b) We make an ultimate non-relativistic approximation for T

$$T \simeq t - \frac{r}{c} \equiv t_e \tag{11.23}$$

Then we derived the radiation field by substituting the current

$$\boldsymbol{J}(t_e) = e\boldsymbol{v}(t_e)\delta^3(\boldsymbol{r}_o - \boldsymbol{r}_*(t_e)) \tag{11.24}$$

into the Eqs. (11.8),(11.9), and (11.17) for the radiated power

(c) The electric field is

$$\boldsymbol{E} = \frac{e}{4\pi r c^2} \boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{a}(t_e) \tag{11.25}$$

Notice that the electric field is of order

$$E \sim \frac{e}{4\pi r} \frac{a(t_e)}{c^2} \tag{11.26}$$

(d) The power per solid angle emitted by acceleration at time t_e is

$$\frac{dP(t_e)}{d\Omega} = \frac{e^2}{(4\pi)^2 c^3} a^2(t_e) \sin^2\theta \tag{11.27}$$

Notice that the power is of order

$$P \sim c|rE|^2 \sim \frac{a^2}{c^3} \tag{11.28}$$

(e) The total energy that is emitted is

$$P(t_e) = \frac{e^2}{4\pi} \frac{2}{3} \frac{a^2(t_e)}{c^3} \tag{11.29}$$

The Electric Dipole approximation

(a) We make the ultimate non-relativistic approximation

$$\boldsymbol{J}(t - \frac{r}{c} + \frac{\boldsymbol{n} \cdot \boldsymbol{r}_o}{c}) \simeq \boldsymbol{J}(t - \frac{r}{c}) \tag{11.30}$$

Leading to an expression for $\boldsymbol{A}_{\rm rad}$

$$\boldsymbol{A}_{\rm rad} = \frac{1}{4\pi r} \frac{1}{c} \partial_t \boldsymbol{p}(t_e) \tag{11.31}$$

where the dipole moment is

$$\boldsymbol{p}(t_e) = \int \mathrm{d}^3 x_o \, \rho(t_e) \boldsymbol{r}_o \tag{11.32}$$

(b) The electric and magnetic fields are

$$\boldsymbol{E}_{\rm rad} = \boldsymbol{n} \times \boldsymbol{n} \times \frac{1}{c} \partial_t \boldsymbol{A}_{\rm rad} \tag{11.33}$$

$$= \frac{1}{4\pi r c^2} \boldsymbol{n} \times \boldsymbol{n} \times \ddot{\boldsymbol{p}}(t_e) \tag{11.34}$$

$$\boldsymbol{B}_{\rm rad} = \boldsymbol{n} \times \boldsymbol{E}_{\rm rad} \tag{11.35}$$

(c) The power radiated is

$$\frac{dP(t_e)}{d\Omega} = \frac{1}{16\pi^2} \frac{\ddot{p}^2(t_e)}{c^3} \sin^2\theta \tag{11.36}$$

(d) For a harmonic source $\boldsymbol{p}(t_e) = \boldsymbol{p}_o e^{-i\omega(t - r/c)}$ the time averaged power is

$$P = \frac{1}{4\pi} \frac{\omega^4}{3c^3} |\boldsymbol{p}_o|^2 \tag{11.37}$$

The magnetic dipole and quadrupole approximation: L32

(a) In the magnetic dipole and quadrupole approximation we expand the current

$$\boldsymbol{J}(T) \simeq \underbrace{\boldsymbol{J}(t_e)}_{\text{electric dipole}} + \underbrace{\frac{\boldsymbol{n}\cdot\boldsymbol{r}_o}{c}\partial_t \boldsymbol{J}(t_e,\boldsymbol{r}_o)/c}_{\text{next term}} \tag{11.38}$$

The next term when substituted into Eq. (11.8) gives rise two new contributions to $\boldsymbol{A}_{\rm rad}$, the magnetic dipole and electric quadrupole terms:

$$\boldsymbol{A}_{\rm rad} = \underbrace{\boldsymbol{A}^{E1}_{\rm rad}}_{\text{electric dipole}} + \underbrace{\boldsymbol{A}^{M1}_{\rm rad}}_{\text{mag dipole}} + \underbrace{\boldsymbol{A}^{E2}_{\rm rad}}_{\text{electric-quad}} \tag{11.39}$$

(b) The magnetic dipole contribution gives

$$\boldsymbol{A}^{M1}_{\rm rad} = \frac{-1}{4\pi r}\frac{\boldsymbol{n}}{c} \times \dot{\boldsymbol{m}}(t_e) \tag{11.40}$$

where $\boldsymbol{m}$

$$\boldsymbol{m} \equiv \frac{1}{2}\int_{\boldsymbol{r}_o} \boldsymbol{r}_o \times \boldsymbol{J}(t_e,\boldsymbol{r}_o)/c\,, \tag{11.41}$$

is the magnetic dipole moment.

(c) The structure of magnetic dipole radiation is very similar to electric dipole radiation with the duality transformation

$$\text{E-dipole} \to \text{M-dipole} \tag{11.42}$$

$$\boldsymbol{p} \to \boldsymbol{m} \tag{11.43}$$

$$\boldsymbol{E} \to \boldsymbol{B} \tag{11.44}$$

$$\boldsymbol{B} \to -\boldsymbol{E} \tag{11.45}$$

(d) The power is

$$\frac{dP^{M1}(t_e)}{d\Omega} = \frac{\ddot{m}^2 \sin^2\theta}{16\pi^2 c^3} \tag{11.46}$$

(e) The power radiated in magnetic dipole radiation is smaller than the power radiated in electric dipole radiation by a factor of the typical velocity, $v_{\rm typ}$ squared:

$$\frac{P^{M1}}{P^{E1}} \propto \frac{m^2}{p^2} \sim \left(\frac{v_{\rm typ}}{c}\right)^2 \tag{11.47}$$

where $v_{\rm typ} \sim L_{\rm typ}/T_{\rm typ}$

Quadrupole rdiation

(a) For quadrupole radiation we have

$$\boldsymbol{A}^{j}_{\rm rad,E2} = \frac{1}{24\pi r}\frac{n_i}{c^2}\dddot{\mathcal{Q}}^{ij} \tag{11.48}$$

where $\mathcal{Q}^{ij}$ is the symmetric traceless quadrupole tensor.

$$\mathcal{Q}^{ij} = \int d^3x_o \rho(t_e,\boldsymbol{r}_o)\,\left(3r^i_o r^j_o - \boldsymbol{r}^2_o\delta^{ij}\right) \tag{11.49}$$

(b) The electric field is

$$\boldsymbol{E}_{\rm rad} = \frac{-1}{24\pi r c^3}\left[\dddot{\boldsymbol{\mathcal{Q}}}\cdot\boldsymbol{n} - \boldsymbol{n}(\boldsymbol{n}^\top\cdot\dddot{\boldsymbol{\mathcal{Q}}}\cdot\boldsymbol{n})\right] \tag{11.50}$$

where (more precisely) the first term in square brackets means $n_i\dddot{\mathcal{Q}}^{ij}$, while the second term means, $(n_\ell\dddot{\mathcal{Q}}^{\ell m}n_m)n^j$.

(c) A fair bit of algebra shows that the total power radiated from a quadrupole form is

$$P = \frac{1}{720\pi c^5} \dddot{\mathcal{Q}}^{ab} \dddot{\mathcal{Q}}_{ab} \tag{11.51}$$

(d) For harmonic fields, $\mathcal{Q} = \mathcal{Q}_o e^{-i\omega t}$, the time averaged power is rises as ω^6

$$P = \frac{c}{1440\pi} \left(\frac{\omega}{c}\right)^6 \mathcal{Q}_o^2 \tag{11.52}$$

(e) The total power radiated radiated in quadrupole radiation to electric-dipole radiation for a typical source size $L_{\rm typ}$ is smaller:

$$\frac{P^{E2}}{P^{E1}} \sim \left(\frac{\omega L_{\rm typ}}{c}\right)^2 \tag{11.53}$$

11.3 Transition to the radiation zone

(a) Starting from the general expression Eq. (11.5), we studied the exact fields of an electric dipole. The current for the dipole is

$$\boldsymbol{J}(t_o, \boldsymbol{r}_o) = \partial_{t_o} \boldsymbol{p}(t_o) \delta^3(\boldsymbol{r}_o) \tag{11.54}$$

$$\boldsymbol{\rho}(t_o, \boldsymbol{r}_o) = -\boldsymbol{p}(t_o) \cdot \nabla_{\boldsymbol{r}_0} \delta^3(\boldsymbol{r}_o) \tag{11.55}$$

Performing the integrals in Eq. (11.5), and differentiating to find the electric and magnetic fields we have

$$\boldsymbol{E}(t, \boldsymbol{r}) = \underbrace{\frac{3(\boldsymbol{n}\cdot\boldsymbol{p}(t_e)) - \boldsymbol{p}}{4\pi r^3}}_{\text{near field}} + \underbrace{\frac{3\boldsymbol{n}(\boldsymbol{n}\cdot\dot{\boldsymbol{p}}(t_e)) - \dot{\boldsymbol{p}}(t_e)}{4\pi r^2 c}}_{\text{intermediate zone}} + \underbrace{\frac{-\ddot{\boldsymbol{p}}(t_e) + \boldsymbol{n}(\boldsymbol{n}\cdot\ddot{\boldsymbol{p}}(t_e))}{4\pi r c^2}}_{\text{radiation field} \propto \, \boldsymbol{n}\times\boldsymbol{n}\times\ddot{\boldsymbol{p}}} \tag{11.56}$$

(11.57)

and

$$\boldsymbol{B}(t, \boldsymbol{r}) = \underbrace{-\frac{\boldsymbol{n}\times\dot{\boldsymbol{p}}(t_e)}{4\pi r^2 c}}_{\text{quasi-static filed}} + \underbrace{-\frac{\boldsymbol{n}\times\ddot{\boldsymbol{p}}(t_e)}{4\pi r c^2}}_{\text{radiation field}} \tag{11.58}$$

(b) The successive terms trade powers of $1/r$ for powers of $1/c\,\partial_t$. The radiation field decreases as $1/r$.

(c) Looking at the electric fields, the first term is the static electric field of a dipole (as we derived in electrostatics), the last term is the radiation field of the static dipole.

(d) Looking at the magnetic field. The first term is what we derived in a quasi-static approximation, and the second term is the radiation field.

(e) The electric and magnetic duality says that the fields of a magnetic dipole can be found with the replacements $\boldsymbol{E} \to \boldsymbol{B}$, and $\boldsymbol{B} \to -\boldsymbol{E}$, $\boldsymbol{p} \to \boldsymbol{m}$

11.4 Attenas

(a) In an antenna with sinusoidal frequency we have

$$\boldsymbol{J}(T, \boldsymbol{r}_o) = e^{-i\omega(t - \frac{r}{c} + \frac{\boldsymbol{n}\cdot\boldsymbol{r}_o}{c})} \boldsymbol{J}(\boldsymbol{r}_o) \tag{11.59}$$

(b) Then the radiation field for a sinusoidal current is:

$$\boldsymbol{A}_{\rm rad} = \frac{e^{-i\omega(t-r/c)}}{4\pi r} \int_{\boldsymbol{r}_o} e^{-i\omega \frac{\boldsymbol{n}\cdot\boldsymbol{r}_o}{c}} \boldsymbol{J}(\boldsymbol{r}_o)/c \tag{11.60}$$

In general one will need to do this integral to determine the radiation field.

(c) The typical radiation resistance associated with driving a current which will radiate over a wide range of frequencies is $R_{\rm vacuum} = c\mu_o = \sqrt{\mu_o/\epsilon_o} = 376\,{\rm Ohm}$.