Last Times (pg. 1)

① $\nabla \cdot E = \rho$

$\nabla \times B = J/c + \frac{1}{c}\partial E/\partial t$

sourced

$\nabla \cdot B = 0$

$-\nabla \times E = \frac{1}{c}\partial_t B$

source free $\nearrow B = \nabla \times A$

$\searrow E = -\frac{1}{c}\partial_t A - \nabla \varphi$

② Waves (in lorentz gauge $\frac{1}{c}\partial_t \varphi + \nabla \cdot A = 0$)

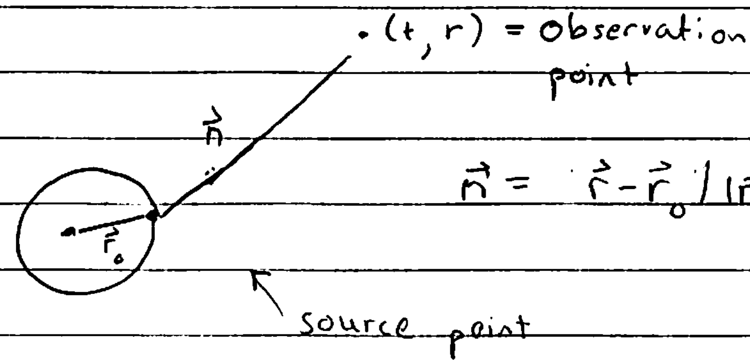
$-\Box \varphi = \rho$

$-\Box \vec{A} = J/c$

③ Solve using green fcn:

$$\vec{A}(t,r) = \int d^3 r_0 \frac{1}{4\pi |\vec{r} - \vec{r}_0|} J(T, r_0)$$

$$T = t - \frac{|\vec{r} - \vec{r}_0|}{c}$$ ← retarded time



$\vec{n} = \vec{r} - \vec{r}_0 / |\vec{r} - \vec{r}_0|$

Last Times pg. 2

At large distances can approximate, even for highly relativistic sources, $r \gg \lambda_{typ} \sim cT_{typ}$

$$\vec{A}_{rad} = \frac{1}{4\pi r} \int \frac{J}{c}\left(t - \frac{r}{c} + \frac{\vec{n}\cdot\vec{r}_0}{c}, r_0\right) d^3 r_0$$

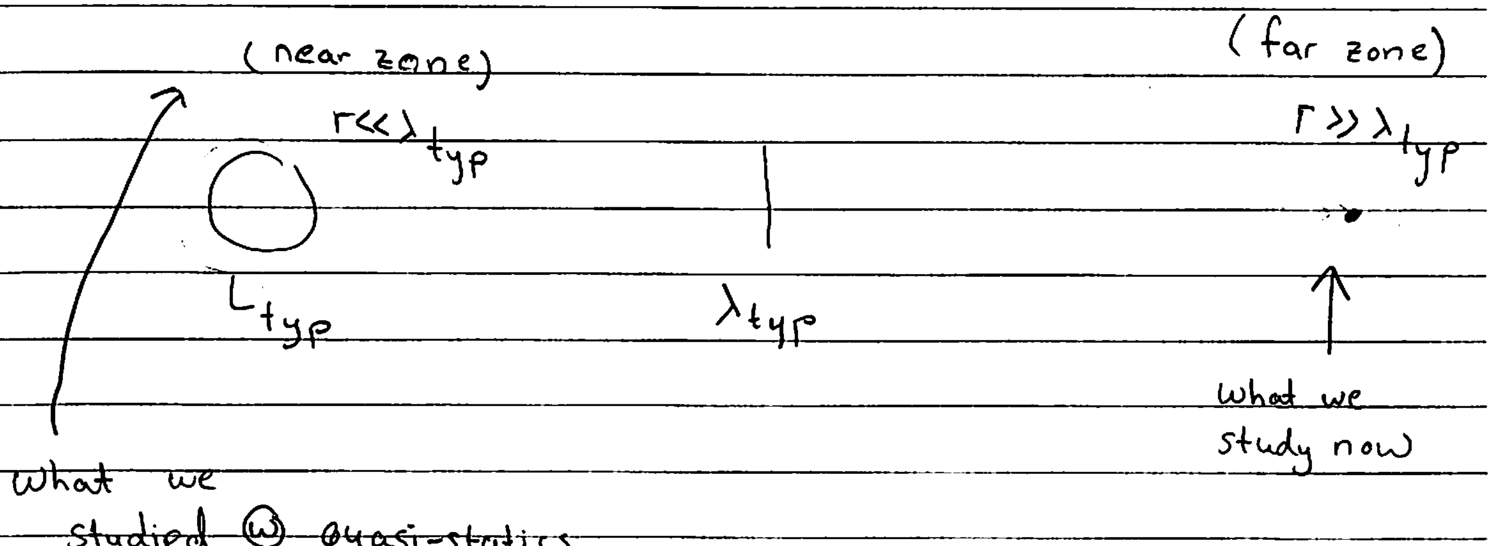
i.e.

$$T = t - \frac{r}{c} + \frac{n \cdot r_0}{c}$$

$$E_{rad} = \vec{n} \times \vec{n} \times \frac{1}{c} \partial_t \vec{A}_{rad} \quad \longleftarrow \quad n \times n \times \vec{J} = -(\text{transverse current})$$

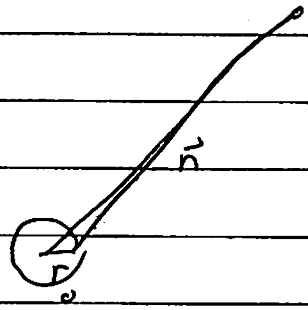
$$B_{rad} = -n \times \frac{1}{c} \frac{\partial A_{rad}}{\partial t}$$

(4) Then we concentrated first on non rel sources where, $L_{typ} \ll cT_{typ}$ or $L \ll \lambda_{typ}$. So for non-rel source, this is the picture



Last Times pg. 3

For non-relativistic sources $n \cdot \vec{r}_0/c$ is small compared to t:



Source

$$T = t - \frac{r}{c} + \frac{n \cdot r_0}{c}$$ small

since $\frac{n \cdot r_0}{c}$ is of order $\frac{L_{typ}}{c} \ll t \sim T_{typ}$.

Thus in a non-relativistic approximation we write:

$$\vec{J}\left(t - \frac{r}{c} + \frac{n \cdot r_0}{c}\right) \simeq J\left(t - \frac{r}{c}\right) + \frac{n \cdot r_0}{c} \frac{\partial J}{\partial t}\left(t - \frac{r}{c}\right) + \ldots$$

So, we define $t_e = t - r/c$ = emission time to save writing:

(5) Two Examples So far:

a) Radiation from a charged particle. In this case one has simply:

$$\int \frac{J(t_e)}{c}\, d^3r = \frac{e\, V(t_e)}{c}$$

and

$$A_{rad} = \frac{e}{4\pi r} \frac{\vec{V}(t_e)}{c}$$

transverse piece of $\vec{a}$

$$E_{rad} = n \times n \times \frac{1}{c}\frac{\partial A_{rad}}{\partial t} = \frac{e}{4\pi r} \frac{-\vec{a}_T(t_e)}{c^2}$$

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a) continued..., Leading to the power radiated

$$P = \int r^2 d\Omega \; c(E \times B)$$

$$\boxed{P = \frac{e^2}{4\pi} \frac{2}{3} \frac{a^2(t_e)}{c^3}}$$

Larmour forula for the power radiated

b) Mutipole expansion of Localized source

6 Multipole Expansion and electric Dipole

$$\vec{J}\left(t - \frac{r}{c} + \frac{n \cdot r_o}{c}\right) = J(t_e) + \frac{n \cdot r_o}{c} \partial_t J + \ldots$$

electric dipole approx

magnetic dipole and quadrupole approx, today

First we had the electric dipole:

dipole moment

$$\vec{A}_{rad} = \frac{1}{4\pi r} \int_{r_o} \frac{J(t_e)}{c} = \frac{1}{4\pi r} \left[\frac{\dot{p}(t_e)}{c}\right]$$

$\vec{J} = \partial_t \vec{P}$ ← capitol $\vec{P}$ is the dipole moment/volume, integrating over volume gives the dipole moment $\vec{p}$

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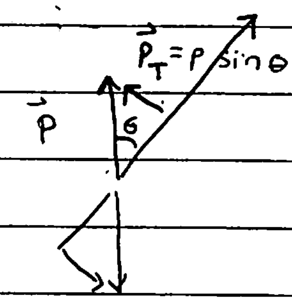
Then, note that this forms a spherical wave.
Take $\vec{p}(t) = p_0 e^{-i\omega t}$, then since we evaluate, $t_e = t - \frac{r}{c}$:

$$\vec{A}_{rad} = -i\omega \frac{p_0 \, e^{-i\omega(t - r/c)}}{4\pi r c}$$ ← outgoing spherical wave.

Then we have:

$$\vec{E}_{rad} = n \times n \times \frac{1}{c}\frac{\partial A_{rad}}{\partial t} = \frac{1}{4\pi r}\left[-\frac{\ddot{\vec{p}}_T(t_e)}{c^2} \right]$$

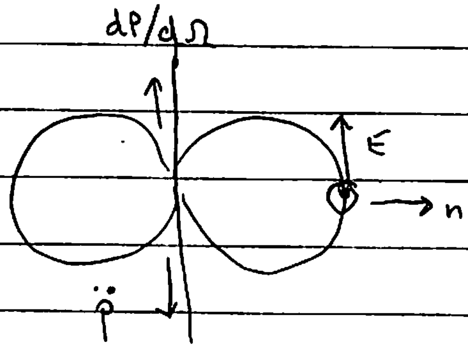
After computing the power, we find for a harmonic source, $p = p_0 e^{-i\omega t}$:



$$\left\langle \frac{dP}{d\Omega} \right\rangle = c\,|r E_{rad}|^2$$

time averaged

$$= \frac{\omega^4}{16\pi^2 c^3} \frac{|p_0|^2}{2} \sin^2\theta$$ ← time average



Then we found a radiation pattern shown to the left. The characteristic features are:

① $P \propto \omega^4$

② $\sin^2\theta$

③ Polarization

Magnetic Dipole (M1) + Electric Quadrupole (E2)

Now we cant contine with the expansion:

$$J\left(t-\frac{r}{c}+\frac{n\cdot r_o}{c}, r_o\right) \simeq \underbrace{J(t_e, r_o)}_{\text{Electric dipole}} + \underbrace{\frac{n\cdot r_o}{c}\frac{\partial J(t_e, r_o)}{\partial t}}_{\text{magnetic dipole and quadrupole}} + \ldots$$

So the next term gives:

$$\vec{A}_{rad} = \frac{1}{4\pi r}\int_{r_o} \frac{\vec{n}\cdot\vec{r}_o}{c}\frac{\partial \vec{J}}{\partial t}(t-r/c, r_o)/c$$

$$A^j_{rad} = \frac{n_i}{4\pi r}\int_{r_o} r_o^i \frac{\partial J^j}{\partial t}(t_e, r_o)/c$$

As always tensors $r_o^i\,\partial J^j/\partial t$ should be broken up into its irreducable components and analyzed separately. We will see that each irred comp gives:

$$r_o^i\,\partial_t J^j = \underbrace{\frac{1}{2}\left(r_o^i\partial_t J^j + r_o^j\partial_t J^i - \frac{2}{3}\delta^{ij}\,\vec{r}_o\cdot\partial_t\vec{J}\right)}_{\text{Quadrupole rad}} + \underbrace{\frac{1}{2}\varepsilon^{ijk}\left(\vec{r}_o\times\partial_t\vec{J}\right)}_{\text{magnetic dipol}} + \underbrace{\frac{1}{3}\vec{r}_o\cdot\partial_t\vec{J}\,\delta^{ij}}_{\text{monopole, gives nothin}}$$

a monopole doesn't radiate

We will first analyze the magnetic dipole case. The general case is a sum of these + e-dipole term

The contribution to $A^j_{\rm rad}$ from this term is proportional n^j (prove me). When computing the E-field we take the piece of $\vec{A}_{\rm rad}$ which is perpendicular to $\vec{n}$

Magnetic Dipole pg 2

For mag-dipole (sign because I reversed i, j relative to last page)

$$A^{j}_{rad} = \frac{1}{4\pi r c}\int_{r_0} -\frac{1}{2}\epsilon^{jik}\, n_i\, (\vec r_0 \times \partial_t \vec J/c)_k$$

$$\vec A_{rad} = -\frac{1}{4\pi r}\,\frac{\vec n}{c}\times\frac{1}{2}\int_{r_0}\vec r_0\times\partial_t\vec J(t_e, r_0)/c$$

$$\vec A_{rad} = -\frac{1}{4\pi r}\,\frac{\vec n}{c}\times\dot{\vec m}(t_e)$$

we defined $\vec m = \frac{1}{2}\int \vec r_0 \times \vec J/c$

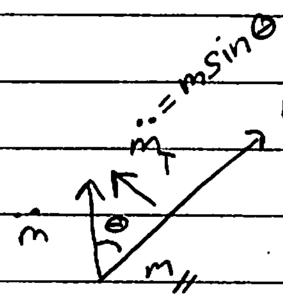
So

$$\vec B_{rad} = -\vec n \times \frac{1}{c}\frac{\partial \vec A_{rad}}{\partial t}$$

Compare this with the electric dipole. E is replaced with B, and $\boldsymbol{p}$ is replaced with $\boldsymbol{m}$

$$= \frac{\vec n \times \vec n}{4\pi r}\times\frac{\ddot{\vec m}(t_e)}{c^2}$$

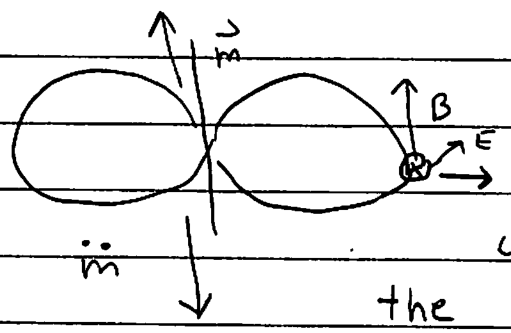
$$= \frac{1}{4\pi r}\left(\frac{-\ddot{\vec m}_T}{c^2}\right)$$



Then the radiated power is:

$$\frac{dP}{d\Omega} = r^2|cE\times B\cdot\vec n|^2 = \frac{\ddot m^2}{16\pi^2 c^3}\sin^2\theta$$

So the angular distribution of power $m(t) = m_0 e^{-i\omega(t - r/c)}$ is the same as the electric case, but the polarization is reversed. This a reflection of Electric-Magnetic duality, which in this context means that the fields of the magnetic dipole are related to the magnetic dipole via the rules:



E-dipole $\longrightarrow$ M-dipole

$$\vec{p} \longrightarrow \vec{m}$$
$$\vec{E} \longrightarrow \vec{B}$$
$$\vec{B} \longrightarrow -\vec{E}$$

Relative strengths of E1 & M1 radiation

- If a system has a magnetic dipole and an electric dipole, then both contribute to the radiation

- Lets compare the size of the two:

$$\vec{p} \sim eL$$

$$\vec{m} \sim \frac{IA}{c} \sim \frac{eL^2}{Tc} \sim eL_{typ}\left(\frac{V}{c}\right)$$

So:

$$\frac{m}{p} \sim \frac{V}{c} \leftarrow \text{small}$$

$$\boxed{\begin{array}{l}\text{And thus the radiated power is smaller for a magnetic dipole by } (V/c)^2 \\ \dfrac{P^{M1}}{P^{E1}} \propto \dfrac{m^2}{p^2} \propto \left(\dfrac{V}{c}\right)^2\end{array}}$$

Quadrupole Radiation

- Now lets compute Quadrupole radiation

The potential fields φ and A are sourced by

Eq.1

$$\frac{1}{2}\left(r_0^i \partial_t J^j + r_0^j \partial_t J^i - \frac{2}{3}\delta^{ij} r_{0\ell}\partial_t J^\ell\right) = \partial_t \mathring{T}^{ij}$$

Using

$$\frac{\partial r_0^j}{\partial r_0^\ell} = \delta^j_\ell \qquad \text{and} \qquad \frac{\partial J^\ell(r_0)}{\partial r_0^\ell} = -\partial_t \rho$$

We have

The second term subtracts off the $\partial J^\ell/\partial r_o^\ell$ term of the first term which we don't have in Eq.1

$$\mathring{T}^{ij} = \frac{1}{2}\frac{\partial}{\partial r^\ell}\left(J^\ell\left(r_0^i r_0^j - \frac{1}{3}\delta^{ij} r_0^2\right)\right) - \underbrace{\frac{\partial J^\ell}{\partial r_0^\ell}}_{-\frac{\partial \rho}{\partial t}}\frac{1}{2}\left(r_0^i r_0^j - \frac{1}{3}r_0^2\delta^{ij}\right)$$

So then

$$A^j_{rad} = \frac{n_i}{4\pi r c}\int_{r_0} \partial_t \frac{\mathring{T}^{ij}}{c}$$

As before, $Q^{ij} \equiv \int d^3r_0(3r_0^i r_0^j - r_0^2\delta^{ij})$

$$= \frac{n_i}{4\pi r c^2}\underbrace{\int_{r_0}\frac{1}{2}\ddot{\rho}(t_e)\left(r_0^i r_0^j - \frac{1}{3}r_0^2\delta^{ij}\right)}_{\equiv \ddot{Q}^{ij}/6}$$

$$A^j_{rad} = \frac{1}{24\pi r c^2} n_i \ddot{Q}^{ij}$$

Or in matrix notation

$$\vec{A} = \frac{1}{24\pi r c^2} \ddot{Q} \cdot n \qquad \vec{A}_T = \vec{A} - \vec{n}(\vec{n}\cdot\vec{A})$$
$$= (\mathbb{1} - nn^T)A$$

$$\vec{E} = -\frac{1}{c}\frac{\partial A_T}{\partial t} = -\frac{1}{24\pi r c^3}(\mathbb{1} - nn^T)\cdot\dddot{Q}\cdot n$$

transvere component of $\vec{A}$

So

$$\vec{E} = -\frac{1}{24\pi r c^3}\left[\dddot{Q}\cdot n - \vec{n}(n^T\cdot\dddot{Q}\cdot n)\right]$$

Now

The is some vector $\boldsymbol{v} \propto \dddot{Q}$. The power is pronto $\boldsymbol{v}\cdot\boldsymbol{v}$

$$\frac{dP}{d\Omega} = c|rE|^2 = \frac{1}{(24\pi)^2 c^5}\left[\dddot{Q}\cdot n - n(n^T\dddot{Q}n)\right]^2$$

Take a specific component to gain inteiition

$$Q^{ij} = \left(\begin{array}{cc|c} -Q_{zz}/2 & & \\ & -Q_{zz}/2 & \\ \hline & & Q_{zz} \end{array}\right) \qquad \text{only } Q_{zz} \text{ specified}$$

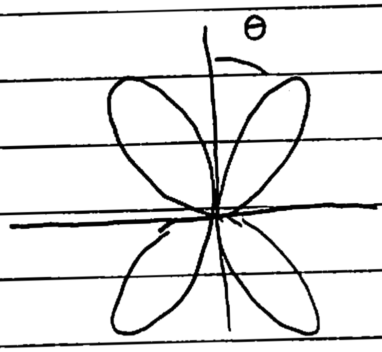
Then take

$$\vec{n} = (\sin\Theta\cos\phi,\ \sin\Theta\sin\phi,\ \cos\Theta)$$

Find for this specific case $\left[\dddot{Q}\cdot n - n(n^T Q n)\right]^2$ = work it out to find

$$\frac{dP}{d\Omega} = \frac{1}{(24\pi)^2 c^5}\left[\frac{9}{16}\dddot{Q}_{zz}^{\,2}\sin^2(2\theta)\right]$$

So we plot



So we see two characteristic lobes associated with Quadrupole radiation.

It is possible to compute the total power is (Homework) in general:

$$P = \int d\Omega \frac{dP}{d\Omega}$$

$$P = \frac{1}{720\pi c^5}\dddot{Q}^{ij}\dddot{Q}_{ij}$$

For harmonic Sources $Q(t) = Q_0 e^{-i\omega t}$, pick up $\frac{1}{2}$ from average over time:

$$\boxed{P = \frac{c}{1440\pi}\left(\frac{\omega}{c}\right)^6 Q_0^{ij} Q^*_{0\,ij}}$$

↳ one sees a characteristic ω^6 dependence

Comparison ⓦ Dipole Radiation

- For dipole radiation, $\overset{\text{e-dipole}}{p} \sim eL$, and

$$\underset{\text{power}}{P} \sim c\left(\frac{\omega}{c}\right)^4 p^2$$

$$\sim c\, e^2 k^2 (kL)^2 \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

- While for Quadrupole radiation, the power is

$$P \sim c\left(\frac{\omega}{c}\right)^6 Q_0^2, \text{ where, } Q_0 \sim eL^2$$

So

$$P \sim c\, e^2 k^2 (kL)^4$$

- So units check:

$$\underbrace{c}_{\text{velocity}} \times \underbrace{e^2 k^2}_{\text{Force}} = \text{Energy}/\text{time}$$

$$\boxed{\begin{array}{l}\text{So we see that quadrupole radiation is suppressed} \\ \text{relative to (Electric) dipole radiation by, } (kL)^2\text{, i.e.} \\ \text{or} \\ \qquad \left(\dfrac{L}{\lambda_{typ}}\right)^2\end{array}}$$