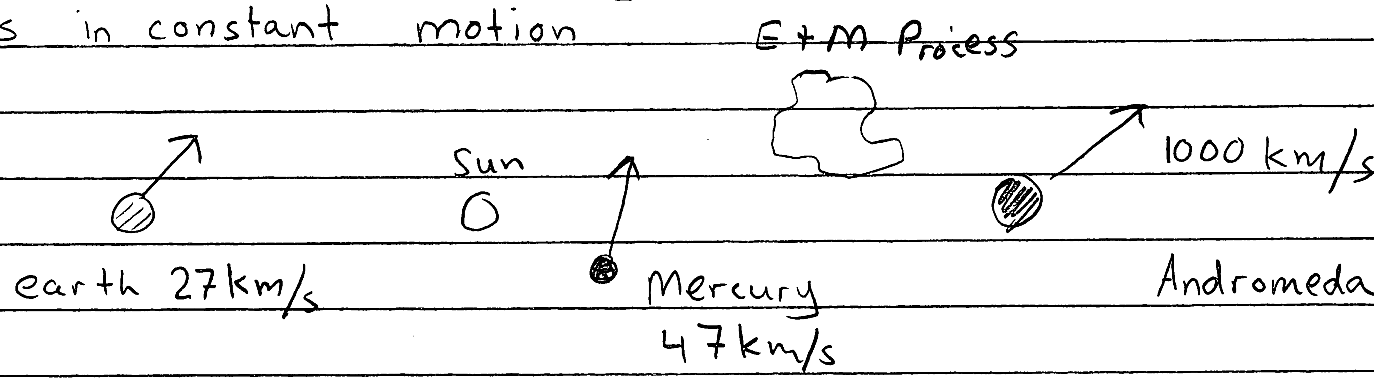
Relativity

Motivation for relativity. The universe is in constant motion



All observers are in constant motion relative to each other. Why is any single observer preferred? Yet they all seem to measure different forces in the E+M processes:

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}_p}{c} \times \vec{B}\right)$$

they all measure a different $\vec{v}_p \equiv$ the particles velocity

Lets write the equations of E+M relativistically:

Earth

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}_p}{c} \times \vec{B}\right)$$

$$-\nabla \cdot E = \rho$$

$$\nabla \times B = \vec{J}/c + \frac{1}{c}\partial_t \vec{E}$$

$$\nabla \cdot B = 0$$

$$-\nabla \times E = \frac{1}{c}\partial_t \vec{B}$$

The only thing relativity changes is the relation between the velocity and momentum:

$$\vec{V}_p = c\,\frac{\vec{p}}{\sqrt{p^2 + (mc)^2}}$$

Then an observer on Andromeda measures the same rules, with his own quantities

$$\underline{t}, \underline{x}, \underline{\vec{p}}, \underline{E}, \underline{B}, \underline{\vec{J}}, \underline{\rho}$$

Andromeda

$$\frac{d\underline{\vec{p}}}{d\underline{t}} = q\left(\underline{\vec{E}} + \frac{\underline{\vec{v}}_p}{c} \times \underline{E}\right)$$

$$-\underline{\nabla}\cdot\underline{E} = \underline{\rho}$$

$$\underline{\nabla}\times\underline{B} = \underline{J}/c + \frac{1}{c}\partial_{\underline{t}}\underline{E}$$

$$\underline{\nabla}\cdot\underline{B} = 0$$

$$-\underline{\nabla}\times\underline{E} = \frac{1}{c}\partial_{\underline{t}}\underline{B}$$

$$\underline{v}_p = \frac{\underline{p}\,c}{\sqrt{(\underline{p}^2 + (mc)^2)}}$$

The speed of Light is constant in all frames/observers.

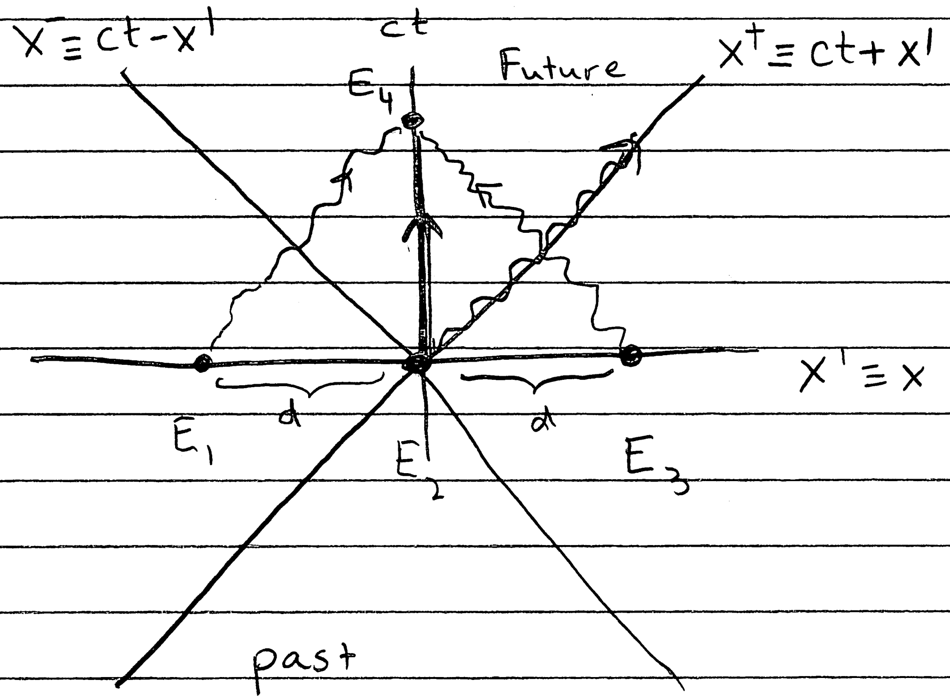
Relativity relates the unbarred quantities (the earth observer) to the barred quantities (the Andromeda observer)

Frames, Events, and Coordinates

Each observer sets up his own coordinate system (K) + labels the events that happen in space time by these coords

$$(x^\mu) = (x^0, x^1, x^2, x^3) \equiv (ct, \vec{x}) \qquad \mu = 0, 1, 2, 3$$

Thus $x^0 \equiv ct$ labels when events happen, while x^i labels where events happen. We can record where and when events happen on a space-time diagram. Consider the following:



Which puts the time (well ct actually) of the events on the y-axis and spatial coordinates on the x-axis

Here we show a number of events.

1. E_1 = Light released at $x = -d$ travelling to right.

2. E_2 = A particle born at rest (thick line), and light born travelling to right along the line, $x^- \equiv ct - x = 0$, with increasing, $x^+ \equiv ct + x$.

3. E_3 = Light born at $x = d$ moving to left (with increasing $x^- \equiv ct - x$ and constant x^+).

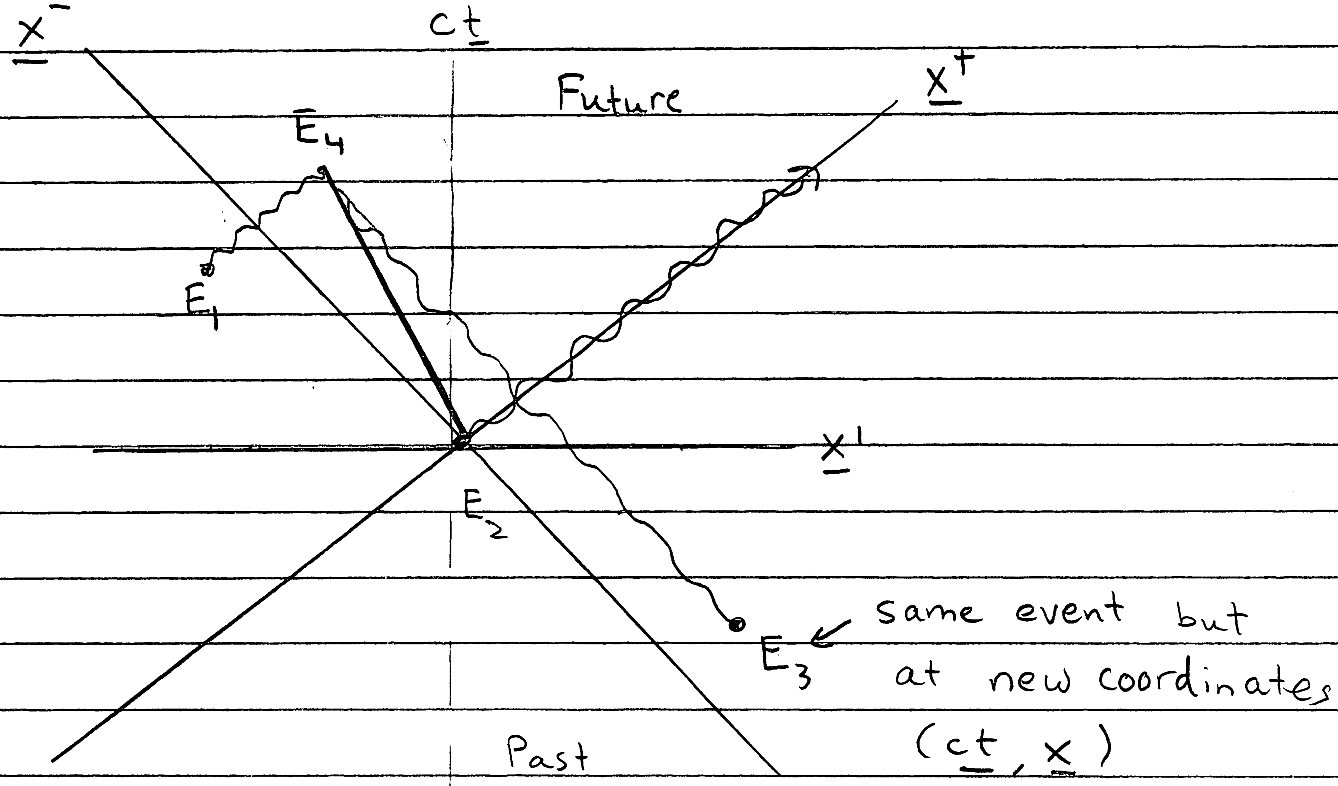
Light rays travel along $45°$ lines with fixed $x^+ \equiv ct + x$ or $x^- \equiv ct - x$. At E_4 the partice & rays meet.

An observer $\underline{K}$ moving to right with velocity v relative to K, measures the particle moving to left vith velocity v. All light rays still move with speed of light = $45°$ lines. The new $\underline{K}$ coordinates are related by Lorentz transformation.

As we will show, the effect of the Lorentz transformation is to (for a boost to the right) contract the x^+ coordinate, and elongate the x^- coordinate for all events. See below:

$$\underline{x}^+ = \sqrt{\frac{1-\beta}{1+\beta}}\, x^+ \qquad \underline{x}^- = \sqrt{\frac{1+\beta}{1-\beta}}\, x^-$$

So the space-time picture looks like in $\underline{K}$:



So in this frame E_3 happens first, then E_2, then E_1. E_4 is causally connected with E_1, E_2, E_3 and therefore must happen after $E_1\, E_2\, E_3$ in all frames. But, E_1, E_2, E_3 are not causally connected and can appear in various orders.

We seek a change of coordinates which leave the trajectory of Light fixed, $c = x/t$

$$-(ct)^2 + x^2 = -\underline{ct}^2 + \underline{x}^2$$

i.e. it is the same for both observers

So $\quad x^\mu \longrightarrow \underline{x}^\mu = L^\mu{}_\nu(v)\, x^\nu$, or as matrices

$$\begin{pmatrix} \underline{x}^0 \\ \underline{x}^1 \\ \underline{x}^2 \\ \underline{x}^3 \end{pmatrix} = \begin{pmatrix} & & & \\ & L(v) & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Properities $\quad L(-\vec{v})L(\vec{v}) = 1 \quad ☆$

$$L(v_2)\,L(v_2) = L(v_3) \quad ☆☆$$

This is known as a group of transformations. The Lorentz Group. With these properties find, for v in x^1 direction

$$\begin{pmatrix} \underline{x}^0 \\ \underline{x}^1 \\ \underline{x}^2 \\ \underline{x}^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}}_{\text{defines } L^\mu{}_\nu} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c}$$

in general use vectors to express boosts in a general direction

Often use a parameter y (the rapidity) to parametrize the boost matrix instead of V to parametrize the boost

$$\frac{V}{c} = \tanh y \rightarrow y = \tanh^{-1} v_p = \frac{1}{2} \ln\left(\frac{1+\beta}{1-\beta}\right)$$

Then $\simeq \beta$ for small β

$\gamma = \cosh y$ and

$\gamma\beta = \sinh y$

The Lorentz boost is a hyperbolic rotation

$$\begin{pmatrix} \underline{x}^0 \\ \underline{x}^1 \end{pmatrix} = \begin{pmatrix} \cosh y & -\sinh y \\ -\sinh y & \cosh y \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

Exercices

① Show that the Lorentz boost compresses x^+ and expands x^-, by the factors of e^{-y} and e^{+y}.

$$\underline{x}^+ = \sqrt{\frac{1-\beta}{1+\beta}}\, x^+ \qquad \underline{x}^- = \sqrt{\frac{1+\beta}{1-\beta}}\, x^-$$

$$\underline{x}^+ = e^{-y} x^+ \qquad \underline{x}^- = e^{+y} x^-$$

Four Vectors and Tensors (Mechanics of Indices)

(1) The coordinates obey a transformation rule

$$\underline{X}^{\mu} = L^{\mu}{}_{\nu} X^{\nu}$$

we call any set of four quantities which transform in this way a four vector. Upper indices are referred to as contravariant components

$$\underline{A}^{\mu} = L^{\mu}{}_{\nu} A^{\nu}$$

$$\underline{B}^{\mu} = L^{\mu}{}_{\nu} A^{\nu}$$

Any two four vectors which transform in this way have an invariant product, $A^{\mu} \equiv (a^0, \vec{a})$ and $B^{\mu} \equiv (b^0, \vec{b})$

$$A \cdot B = -a^0 b^0 + \vec{a} \cdot \vec{b} = \underline{A} \cdot \underline{B} = -\underline{a}^0 \underline{b}^0 + \underline{\vec{a}} \cdot \underline{\vec{b}}$$

because L was adjusted to preserve this quadratic form

(2) For any set of (upper) contravariant indices, define their (lower) covariant counterparts with the metric tensor:

$$X_{\mu} = (-ct, \vec{x})$$

$$X_{\mu} = g_{\mu\nu} X^{\nu} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Lowering indices just changes sign of 0-th component, Sim; we raise indices with $g^{\mu\nu}$:

$$X^\mu = g^{\mu\nu} X_\nu \qquad g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

So clearly

$$X^\mu = \underbrace{g^{\mu\nu} g_{\nu\sigma}}_{\delta^\mu{}_\sigma = g^\mu{}_\sigma \,=\, \text{Identity matrix}} X^\sigma \qquad g^\mu{}_\sigma = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

We define covariant indices so the inner product, can now be written as

$$A \cdot B = A_\mu B^\mu = -a^0 b^0 + \vec{a} \cdot \vec{b}$$

(3) Note if $A_\mu A^\mu$ is to be invariant under Lorentz transformation, then since $\underline{A}^\mu = L^\mu{}_\nu A^\nu$, we define the transformation rule for lower indices (covariant components) with L^{-1} and as a row

$$\underline{A}_\nu = A_\mu (L^{-1})^\mu{}_\nu \quad \text{or} \quad \underline{A}_\nu = (L^{-1T})_\nu{}^\mu A_\mu$$

i.e lower indices transform with inverse and as a row

$$(\underline{A}_0, \underline{A}_1, \underline{A}_2, \underline{A}_3) = (A_0\ A_1\ A_2\ A_3) \begin{pmatrix} L^{-1} \end{pmatrix}$$

So that

$$\underline{A}_\mu \underline{B}^\mu = (\underline{A}_0\ \underline{A}_1\ \underline{A}_2\ \underline{A}_3) \begin{pmatrix} \underline{B}^0 \\ \underline{B}^1 \\ \underline{B}^2 \\ \underline{B}^3 \end{pmatrix}$$

$$= (A_0\ A_1\ A_2\ A_3) \left(L^{-1}\right) \left(L\right) \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix}$$

$$= A_\mu B^\mu$$

(4) Note that under lorentz transform

$$\underline{A} \cdot \underline{B} = \underline{A}^\mu g_{\mu\nu} \underline{B}^\mu = -\underline{a}^0 \underline{b}^0 + \underline{\vec{a}} \cdot \underline{\vec{b}}$$

$$= A^\mu g_{\mu\nu} B^\nu = -a^0 b^0 + \vec{a} \cdot \vec{b} = A \cdot B$$

Without the need of transforming $g_{\mu\nu}$. This says that $g_{\mu\nu}$ is an invariant tensor

$$(L^{-1T})_\mu{}^\rho \, g_{\rho\sigma} \, L^{-1\sigma}{}_\nu = g_{\mu\nu}$$

Or in matices:

$$(L^{-1T}) \, g \, L^{-1} = g$$

i.e.

$$L^{-1T} = g L g \qquad (\text{note } g^{-1} = g)$$

Restoring indices

$$g_{\mu\nu}(L^{\nu}{}_{\rho})\, g^{\rho\sigma} = (L^{-1T})_{\mu}{}^{\sigma}$$

In a fit of notational madness (which is standard) we define

$$\boxed{L_{\mu}{}^{\sigma} \equiv g_{\mu\nu} L^{\nu}{}_{\rho}\, g^{\rho\sigma} = (L^{-1T})_{\mu}{}^{\sigma}}$$

So perhaps its not so mad...

$$\underline{A}_{\mu} = A_{\mu}(L^{-1})^{\mu}{}_{\nu}$$

$$= (L^{-1T})_{\nu}{}^{\mu} A_{\mu}$$

$$\boxed{\underline{A}_{\mu} = L_{\nu}{}^{\mu} A_{\mu}}$$

Excercise

- A tensor transforms as

$$\underline{T}^{\mu\nu} = L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} T^{\rho\sigma}$$

Show that the transformation rule for $T^{\mu}{}_{\nu}$ is

$$\underline{T}^{\mu}{}_{\nu} = L^{\mu}{}_{\rho} T^{\rho}{}_{\sigma} (L^{-1})^{\sigma}{}_{\nu}$$

or equivalently

$$\underline{T}^{\mu}{}_{\nu} = L^{\mu}{}_{\rho} L_{\nu}{}^{\sigma} T^{\rho}{}_{\sigma}$$

Solution:

- Lower ν,

$$T^{\mu}{}_{\nu} = L^{\mu}{}_{\rho} L_{\nu\sigma} T^{\rho\sigma}$$

$$T^{\mu}{}_{\nu} = L^{\mu}{}_{\rho} L_{\nu}{}^{\sigma} T^{\rho}{}_{\sigma}$$

$$= L^{\mu}{}_{\rho} T^{\rho}{}_{\sigma} (L^{-1T})_{\nu}{}^{\sigma}$$

$$= L^{\mu}{}_{\rho} T^{\rho}{}_{\sigma} (L^{-1})^{\sigma}{}_{\nu}$$

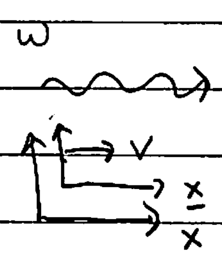
Excercise:

- Explain that if a phane wave of Light $e^{-i\omega t + \vec{k}\cdot\vec{x}}$ where $k = \frac{\omega}{c}$, is to move at the speed of Light in all frames, then

$$K^{\mu} = \left(\frac{\omega}{c}, \vec{k}\right)$$

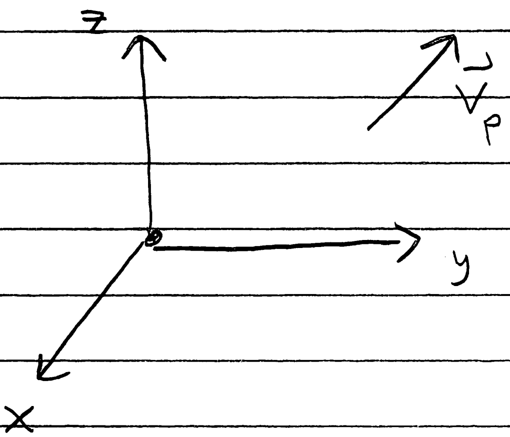
must be a four vector. (1) Show that $K \cdot K = 0$. (2) and Show the relativistic doppler shift formula is

$$\underline{\omega} = \sqrt{\frac{1-\beta}{1+\beta}}\;\omega$$

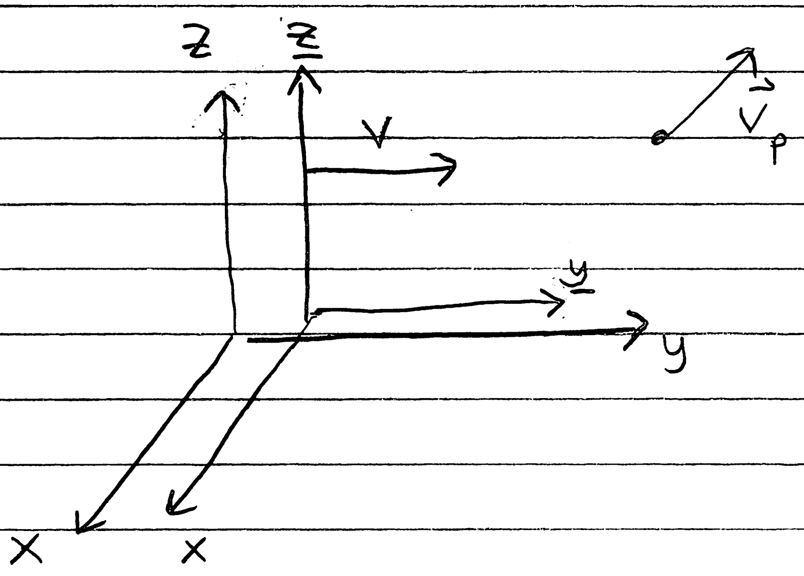


Particles In Special Relativity

Consider a particle with velocity $\vec{\beta}_p = \vec{V}_p / c$. Note we put the "p" sub-label to keep it apart from β and v which will label the velocity of a new frame/observer that we wish to boost to.



We will calculate $\underline{V}_p$ in another frame below. The new frame moves with velocity $\vec{V}$ relative to the first, and has its own coordinate $(c\underline{t}, \underline{x}, \underline{y}, \underline{z})$ shown below



- The change in coordinates of the partcle is

$$dx^\mu = (c\,dt, d\vec{x}) = \underbrace{dx^0}_{=c\,dt}(1, \vec{\beta}_p)$$

where

$$\vec{\beta}_p = \frac{d\vec{x}}{dx^0} = \frac{\vec{v}_p}{c}$$

Then we can construct the invariant space-time interval

$$ds^2 = dx_\mu dx^\mu = -c^2 dt^2 (1 - \beta_p^2) \quad (\star)$$

Since it is invariant we can interpret it in any frame. In the rest frame of the particle, the particle is not moving, $\beta_p = 0$, its time increases:

$$dt \text{ in Local Rest Frame} \equiv d\tau \equiv \text{propper time}$$

So

$$\boxed{ds^2 = -c^2 d\tau^2}$$

And

$$d\tau = \frac{\sqrt{-ds^2}}{c} = \frac{1}{c}\sqrt{-dx_\mu dx^\mu}$$

Then we have from Eq ☆

$$d\tau = dt\sqrt{1-\beta_p^2}$$

$$\boxed{d\tau = \frac{dt}{\gamma_p}} \qquad \gamma_p \equiv \frac{1}{\sqrt{1-\beta_p^2}}$$

Then we define the four velocity u^μ as

$$\boxed{u^\mu \equiv \frac{dx^\mu}{d\tau}}$$

Now

$$dx^\mu = dt\left(c, \frac{d\vec{x}}{dt}\right), \qquad \text{or}$$

$$dx^\mu = \gamma_p\, d\tau\, (c, \vec{v}_p).$$

So

$$\boxed{u^\mu = (\gamma_p c, \gamma_p \vec{v}_p)} \qquad (☆☆)$$

Clearly $u^\mu u_\mu = -c^2$

$$u^\mu u_\mu = \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = \frac{ds^2}{(d\tau)^2} = -c^2$$

You can also verify directly from Eq (☆☆)

Energy + Momentum

Consider the action of a particle

$$S = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} (p\dot{q} - H(p,q))\,dt$$

So for a free particle $\vec{p} = \text{const}$ $H = E$ and we have

$$S = \int p\,d\vec{x} - E\,dt$$

So since $(c\,dt, d\vec{x})$ is a four vector it is theoretically tempting to define a four vector

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right)$$

Then $P \cdot dX = P_\mu dx^\mu = -E\,dt + \vec{p}\cdot d\vec{x}$ is lorentz invariant. Requiring at small velocities $\vec{p} = m\vec{v}_p$ leads to

$$p^\mu = mU^\mu = (mc\gamma_p, m\gamma_p v_p) = \left(\frac{E}{c}, \vec{p}\right)$$

in particular we see that the energy of a particle is $\boxed{E = \gamma_p mc^2}$

Energy and Momentum Conservation

Consider the reaction $1 + 2 \longleftrightarrow 3 + 4$

Energy and momentum are conserved

$$P_1^\mu + P_2^\mu = P_3^\mu + P_4^\mu \qquad (\star)$$

In addition the energies of each of the particles are related to their masses.

In working with collisions we use a shorthand notation setting $c = 1$. With this we have, for example:

$$P^2 = P_\mu P^\mu = -m^2, \quad E = \sqrt{p^2 + m^2},$$

$$V_p = \frac{\vec{P}}{E} \quad \text{etc.}$$

One constraint on the reaction kinematics is found by squaring Eq. $\star$, e.g.

$$(P_1 + P_2)^2 = P_1^2 + P_2^2 + 2P_1 \cdot P_2$$

$$= -m_1^2 - m_2^2 + (-2E_1E_2 + p_1p_2\cos\theta)$$

So

Or squaring both sides

$$m_1^2 + m_2^2 + 2E_1E_2 - \vec{p}_1 \cdot \vec{p}_2 =$$

$$m_3^2 + m_4^2 + 2E_3E_4 - \vec{p}_3 \cdot \vec{p}_4$$

This is only one constraint. In general one needs to work fairly hard, writing (Eq ✱)

$$E_1 + E_2 = E_3 + E_4$$

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4$$

in components and solving the equations to find the relations between energy and momenta between the particles.