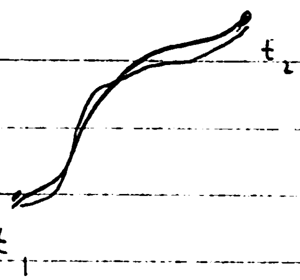
A primer on actions

- Todays Lecture will derive EOM fron actions



$$I_0 = \int dt \, \frac{1}{2} m \dot{x}^2 \qquad I_{int} = \int F_0 \, dx$$

The EOM are derived by varying the path $x(t) \to x(t) + \delta x(t)$ and demanding that the action be stationary

$$\delta I = \delta I_0 + \delta I_{int}$$

$$= \int dt \, \delta x(t) \left[-m\ddot{x} + F \right]$$

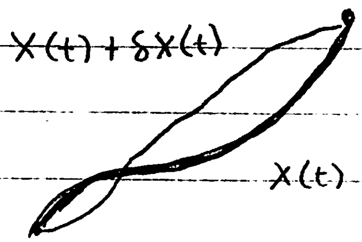
So EOM are

$$\underbrace{m\ddot{x}}_{-\frac{\delta I_0}{\delta x}} = \underbrace{F}_{\frac{\delta I_{int}}{\delta x}}$$

$\frac{dp}{dt} \rightarrow -\frac{\delta I_0}{\delta x}$ $\qquad \frac{\delta I_{int}}{\delta x} \leftarrow$ Force

Keep this in mind today

Super Slow Dissussion of Variations and Variational Derivs



Vary the path

$$x(t) \rightarrow x(t) + \underbrace{\delta x(t)}_{\text{small; vanishes at end}}$$

Then

$$\frac{1}{2} m (\dot{x} + \delta \dot{x})^2 = \frac{1}{2} m \dot{x}^2 + m \dot{x}\, \delta \dot{x} + O(\delta x^2)$$

Now as $x(t)$ is changed the integral changes, $I_{tot} \rightarrow I_{tot} + \delta I$

$$I_{tot} \rightarrow I_{tot} + \delta I = \int \frac{1}{2} m (\dot{x} + \delta \dot{x})^2 + F(x + \delta x)\, dt$$

$$= I_{tot} + \int dt \left[m \dot{x}\, \delta \dot{x} + F \delta x \right]$$

by parts

$$= I_{tot} + \int dt \left[-\frac{d}{dt}(m \dot{x}) + F \right] \delta x(t)$$

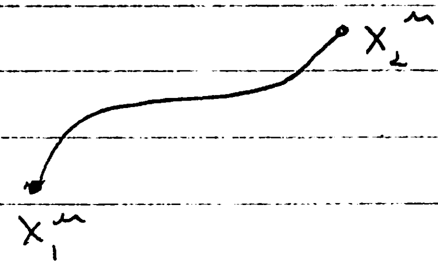
Now $\delta x(t)$ is arbitary. The only way to guarrantee that $\delta I_{tot} = 0$ is if the thing in square brackets vanishes.

i.e. The thing in front of $\delta x(t)$ is known as the variational derivative at time t

$$\frac{\delta I_{tot}}{\delta x(t)} = -\frac{d}{dt}(m \dot{x} + F)$$

The point particle Action pg. 1

For a point particle



$S = \int$ all possible forms allowed by Lorentz invariance and gauge invariance

Parametrize path by $x^\mu(\rho)$ ← parameter

Then

$$-\frac{dx^\mu}{d\rho}\frac{dx_\mu}{d\rho} \quad \text{is invariant}$$

Action should also not depend on paremetrizatn:

$$\frac{dx^\mu}{d\rho} \longrightarrow \frac{d\rho'}{d\rho}\frac{dx^\mu}{d\rho'}$$

Point Particle pg. 2

This restricts the action to:

$$I_0 = k \int dp \sqrt{-\frac{dx^\mu}{dp} \frac{dx_\mu}{dp}}$$

Some const, later take it to be $-mc$

Often take p to be the propper time. Then

$$I_0 = -mc^2 \int d\tau \qquad \text{where } c\,d\tau = \sqrt{-dx_\mu dx^\mu}$$

So we used

$$= dp \sqrt{\frac{-dx^\mu}{dp} \frac{dx_\mu}{dp}}$$

$$d\tau = \frac{dt}{\gamma_p} = dt \sqrt{1 - \dot{x}^2/c^2}$$

and thus, in the non-rel limit,

mc chosen so KE is $\frac{1}{2}mv^2$

$$L_0 = -mc^2 \left(1 - (\dot{x}/c)^2\right)^{-1/2} \simeq -mc^2 + \frac{1}{2} m \dot{x}^2$$

Now

$$\delta I_0 = -\delta \int dt \; mc^2 \sqrt{1 - \dot{x}^2/c^2}$$

$$= \int dt \; \frac{m \dot{x}}{\sqrt{1 - \dot{x}^2}} \partial_t \delta x$$

$$= - \int dt \; \delta x \left[\frac{d}{dt} \left(\gamma m v \right) \right]$$

So the EOM in absence of interactions

$$\frac{d}{dt}(\gamma m \vec{v}) = 0$$

Covariant Formulation of Point Particle

$$I_0 = -\int mc \sqrt{-\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}}\, d\lambda \qquad c\,d\tau = \sqrt{-\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}}\, d\lambda$$

Where we parametrize the path @ λ. So the variation $x^\mu \rightarrow x^\mu + \delta x^\mu$ gives

$$\delta I_0 = \int \frac{-mc}{\left(-\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}\right)^{+1/2}} \cdot -\frac{dx^\mu}{d\lambda}\frac{d\,\delta x_\mu}{d\lambda}$$

integrate by parts

Then

$$\star) \quad \delta I_0 = -\int d\lambda \left[\frac{d}{d\lambda}\frac{mc}{\sqrt{-\dot{x}\cdot\dot{x}}}\frac{dx^\mu}{d\lambda}\right]\delta x_\mu$$

In terms of propper time:

$$d\lambda\,\frac{d}{d\lambda} = d\tau\,\frac{d}{d\tau} \quad \text{with} \quad \frac{d}{d\tau} = \frac{c}{\sqrt{-\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}}}\frac{d}{d\lambda}$$

So Eq $\star$ reads:

$$\delta I_0 = -\int d\tau \left[\frac{d}{d\tau}\left(m\frac{dx^\mu}{d\tau}\right)\right]\delta x_\mu$$

So in absence of interactions, EOM are $m\frac{d^2x}{d\tau^2} = 0$

The interaction lagrangian

$$I_{int} = \frac{e}{c}\int d\lambda \, \frac{dx^\mu}{d\lambda} A_\mu \leftarrow \text{invariant under reparametrization}$$

only Lorentz invariant linear in fields $F_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$

or

$$= \frac{e}{c}\int d\tau \, \frac{dx^\mu}{d\tau} A_\mu$$

$d\tau = \frac{dt}{\gamma}$ $\quad \frac{dx^\mu}{d\tau} = (\gamma c, \gamma \vec{v})$

or

$$\boxed{I_{int} = \int dt \left[-e\varphi + \frac{d\vec{x}}{dt}\frac{\vec{A}}{c} \right]}$$

most usual form in non-rel context

So, $A^\mu \doteq A^\mu(x(\lambda))$

$$\delta I_{int} = \frac{e}{c}\int d\lambda \, \frac{d\,\delta x^\mu}{d\lambda} A_\mu + \frac{dx^\mu}{d\lambda}\frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu$$

$$= \frac{e}{c}\int d\lambda \, \delta x^\mu \left(-\frac{\partial A_\mu}{\partial \lambda}\right) + \frac{dx^\mu}{d\lambda}\frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu$$

Using

$$\frac{\partial A_\mu}{\partial \lambda} = \frac{\partial A_\mu}{\partial x^\rho}\frac{dx^\rho}{d\lambda}$$

Re labelling contracted indices

$$\delta I_{int} = \int d\lambda \left[\frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha} \right] \frac{dx^\alpha}{d\lambda} \delta x^\beta$$

$$\delta I_{int} = \int d\lambda \; F_{\beta\alpha} \frac{dx^\alpha}{d\lambda} \delta x^\beta$$

$$dp \frac{d}{dp} = d\tau \frac{d}{d\tau}$$

$$= \int d\tau \; F_{\beta\alpha} \underbrace{\frac{dx^\alpha}{d\tau}}_{\equiv u^\alpha} \delta x^\beta$$

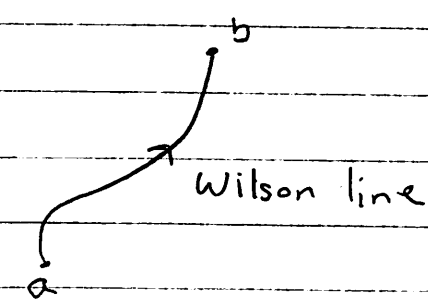
So then

$$\delta I_{TOT} = \delta I_0 + \delta I_{int}$$

$$= \int d\tau \left[-m \frac{d^2 x}{d\tau^2} + F_{\beta\alpha} u^\alpha \right] \delta x^\beta$$

Summary ⓦ view to quantum mechanics

$$e^{iS[x]} = e^{i \frac{e}{c} \int_a^b dx_\mu A^\mu} \longrightarrow$$



Hamiltonian Formulation

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\varphi + \frac{ev}{c}\cdot\vec{A}$$

To construct the hamiltonian we first construct the canonical momentum

$$P_{can} = \frac{\delta S}{\delta \dot{x}} = \frac{\partial L}{\partial v} = \boxed{\frac{m\vec{v}}{\sqrt{1-v^2/c^2}} + \frac{e\vec{A}}{c} = \vec{p}_{kin} + \frac{e\vec{A}}{c} = \vec{P}_{can}}$$

Here $\vec{p}_{kin} = m\gamma\vec{v}$ is the kinetic momenta. Sometimes I will stop writing "kin". Then

$$H = P_{can}\cdot v - L$$

do it!

$$H = \frac{mc^2}{\sqrt{1-\beta^2}} + e\varphi$$

Now for the Hamiltonian framework one should express β in terms of P_{can}. Using

$$\left(P_{can} - \frac{e}{c}A\right)^2 = -(mc)^2 + \frac{(mc)^2}{1-\beta^2}$$

We have

$$\boxed{H = c\sqrt{\left(P_{can} - \frac{eA}{c}\right)^2 + (mc)^2} + e\varphi}$$

In the non-rel limit $mc \gg P_{can} - \frac{eA}{c}$ and find

$$H \simeq mc^2 + \frac{\left(P_{can} - \frac{e}{c}A\right)^2}{2m} + e\varphi$$

This is much easier to derive from the non-relativistic Lagrangian:

$$L = \frac{1}{2}mv^2 - e\varphi + \frac{e\vec{V}}{c}\cdot\vec{A}$$

$$P_{can} = mV + \frac{eA}{c}$$

$$H = p\dot{q} - L = \boxed{\frac{\left(P_{can} - \frac{e}{c}\vec{A}\right)^2}{2m} + e\varphi = H}$$