E Separated coordinates for magnetostatics

E.1 Overview

(a) The magnetostatic equations are complicated, and we refer to Wikipedia for the form of the vector Laplacian in various coordiante systems.

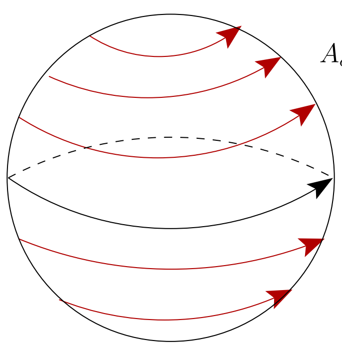
(b) For currents running up stricly up and down $A_z(x, y)$ the magneto static equations reduce to

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) A_z(x, y) = 0 \tag{E.1}$$

which has the same form as $2D$ electrostatics. The appropriate separated solutions are given in Appendix D.4.

(c) For currents which are azimuthally symmetric $\boldsymbol{j} = j_\phi(r, \theta)\,\widehat{\boldsymbol{\phi}}$ we may either use spherical or cylindrical coordinates. The spherical case is discussed in Appendix E.2.

E.2 Spherical coordinates for magnetostatics



(a) The vector Laplacian for azimuthally symmetric currents, and the ansatz $\boldsymbol{A} = A_\phi(r, \theta)\,\widehat{\boldsymbol{\phi}}$ reads

$$\left[-\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\right] A_\phi(r, \theta) = 0 \tag{E.2}$$

This is an appropriate equation only if the current takes a specific symmetric form

$$\boldsymbol{j} = j_\phi(r, \theta)\,\widehat{\boldsymbol{\phi}}$$

(b) The eigen fucntions are along the boundary direction θ, and are regular at $\theta = 0$ and π. They are associated Legendre Polynomials with $m = 1$

$$\psi_\ell(\theta) = P_\ell^1(\cos\theta) \qquad \ell = 1 \ldots \infty$$

The first few eigenfunctions are given here. Perhaps the most important fact is that they all are proportional to $\sin(\theta)$ guaranteeing regularity of $\boldsymbol{A} = A_\phi \,\widehat{\boldsymbol{\phi}}$ at $\theta = 0$ and π.

(c) Orthogonality:

$$\int_{-1}^{1} d(\cos\theta)\; P_\ell^1(\cos\theta)\; P_{\ell'}^1(\cos\theta) = \frac{2}{2\ell+1}\frac{(\ell+1)!}{(\ell-1)!}\delta_{\ell\ell'}$$

(d) Completeness

$$\sum_{\ell=1}^{\infty} \frac{2\ell+1}{2}\frac{(\ell-1)!}{(\ell+1)!} P_\ell^1(x) P_\ell^1(x') = \delta(x - x') \tag{E.3}$$

(e) Solution

$$A_\phi(r,\theta) = \sum_{\ell=1}^{\infty} \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] P_\ell^1(\cos\theta) \tag{E.4}$$