

Vectors and Tensors

- We will use a new notation for vectors, which is very common,

$$\vec{v} = v^1 \vec{e}_1 + v^2 \vec{e}_2 + v^3 \vec{e}_3 = \sum_{i=1}^3 v^i \vec{e}_i$$

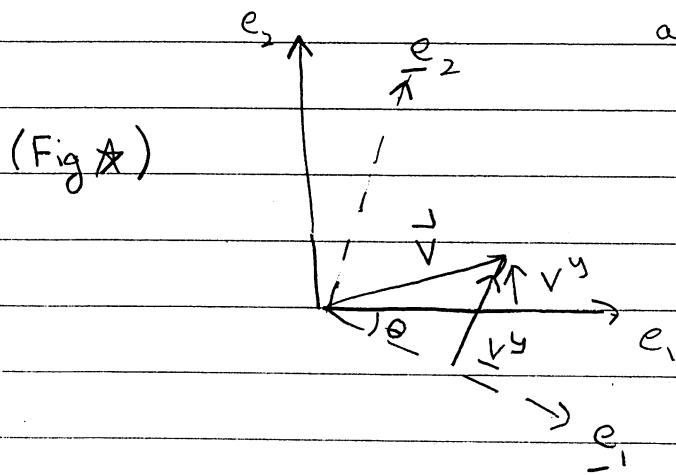
Where $(v^x, v^y, v^z) = (v^1, v^2, v^3)$ and $(\hat{i}, \hat{j}, \hat{k}) = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$

Then we use a summation convention, where repeated indices are summed from 1 to 3.

$$\vec{v} = v^i \vec{e}_i \quad (\text{same as } \vec{v} = \sum v^i \vec{e}_i)$$

- Vectors are physical objects:

If the coordinates are rotated \vec{v} remains unchanged. But the components v^i are changed (see figure, which shows how v^y is changed), and the basis vectors \vec{e}_i are also changed (rotated)



$$v'^i = R^i_j v^j$$

↑ rotated vector components ↑ rotation matrix ← original vector components

We use the summation convention here.

Think of v^i as a column: $\begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix} = \begin{pmatrix} v^i \\ | \\ | \end{pmatrix}$

Then the rotated vector is

$$\begin{pmatrix} v^i \\ | \\ | \end{pmatrix} = \begin{matrix} i & j \\ \downarrow & \rightarrow \end{matrix} \begin{pmatrix} R^i_j \\ | \\ | \end{pmatrix} \begin{pmatrix} v^j \\ | \\ | \end{pmatrix}$$

where, for the rotation shown in Fig★, R is:

$$(R^i_j) = \begin{matrix} i & j \\ \downarrow & \rightarrow \end{matrix} \begin{pmatrix} \cos\theta & -\sin\theta & | & 0 \\ \sin\theta & \cos\theta & | & 0 \\ 0 & 0 & | & 1 \end{pmatrix}$$

Note: Rotations don't change norm. So

$$\underline{v^T v} = v^T \underbrace{R^T R}_I v = v^T v$$

So: R is orthogonal, $R^{-1} = R^T$

- Then, since \vec{v} is unchanged under rotation, we need that the basis vectors transform as the inverse transformation and as a row:

$$\underline{\vec{e}}_i = \underline{\vec{e}}_j (R^{-1})^j_i$$

↑ rotated basis vector
↑ original basis

Think of basis vectors as a row

$$(\underline{e}_1, \underline{e}_2, \underline{e}_3) = (e_1, e_2, e_3) (R^{-1})$$

In this way, \vec{v} is unchanged under rotation:

$$\begin{aligned} \underline{\vec{v}} &= \underline{\vec{e}}_i v^i \\ &= (e_i, \dots) \underbrace{(R^{-1})(R)}_{\mathbb{1}} \begin{pmatrix} v^1 \\ \vdots \\ v^i \end{pmatrix} \\ &= \underline{\vec{e}}_j v^j = \underline{\vec{v}} \end{aligned}$$

We used

$$(R^{-1})^i_j R^j_k = \delta^i_k \equiv \begin{cases} \text{identity matrix} & = 1 \text{ if } i=k \\ & 0 \text{ otherwise} \end{cases}$$

Contravariant / Covariant indices

For every set of upstairs indices (contravariant) (v^x, v^y, v^z) define the lowered (covariant) components (v_x, v_y, v_z) which transform as a row according to the inverse matrix

$$\underline{v}_i = v_j (R^{-1})^j_i$$

$$(\underline{v}_i, \dots) = (v_j, \dots) \begin{pmatrix} R^{-1} \end{pmatrix}$$

Now since $R^{-1} = R^T$ (since rotations preserve length i.e. $\underline{x}^T \underline{x} = x^T x$), we see that

$$v_i = v_j (R^T)^j_i = v_j (R)_j^i$$

i.e.

$$\underline{v}_i = (R)_j^i v_j$$

But this is the same transformation rule as for upstairs indices. So up and down are the same for rotations.

$$v_x = v^x, \quad \text{or} \quad v_i = \delta_{ij} v^j \quad v^i = \delta^{ij} v_j$$

So indices are raised and lowered with δ^{ij} , + δ_{ij}

Covariant Contravariant pg. 2

Similarly define contravariant basis vectors

$$\vec{e}^i = \delta^{ij} \vec{e}_j$$

Which transform as a column vector

$$\vec{e}^i = R^i_j e^j$$

So that the vector is rotationally invariant

$$\vec{v} = v_i \vec{e}^i = v^i \vec{e}_i$$

Dot-Products and Cross-Products:

$$\begin{aligned} \textcircled{1} \quad \vec{a} \cdot \vec{b} &= (a^i \vec{e}_i) (b^j \vec{e}_j) = a^i b^j \overbrace{\vec{e}_i \cdot \vec{e}_j}^{= \delta_{ij}} \\ &= a^i b^j \delta_{ij} = a_i b^i \end{aligned}$$

Rotationally invariant. Prf. easy. Contracted indices are invariant.

② To define cross product, need the epsilon tensor:

$$\epsilon^{ijk} = \begin{cases} \pm 1 & \text{for } ij, k \text{ even/odd permutation} \\ & \text{of } 1, 2, 3 \\ 0 & \end{cases}$$

$$\text{e.g. } \epsilon^{123} = \epsilon^{312} = \epsilon^{231} = \epsilon^{123} = -\epsilon^{213} = -\epsilon^{321} = +1$$

Then this is a tensor which is the same after rotation. Prf uses $\det R = 1$.

$$\det R = R^i{}_i R^j{}_j R^k{}_k \epsilon^{ijk}$$

$$= \begin{vmatrix} R^1{}_i & \dots & \dots \\ R^2{}_j & \dots & \dots \\ R^3{}_k & \dots & \dots \end{vmatrix}$$

Cross Products pg. 2

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{vmatrix} = \epsilon_i \epsilon^{ijk} a_j b_k$$

So

$$(\vec{a} \times \vec{b})^i = \epsilon^{ijk} a_j b_k = \text{i-th } \overset{\text{contravariant}}{\vee} \text{ component of } \vec{a} \times \vec{b}$$

The

$$\begin{aligned} (\vec{a} \times (\vec{b} \times \vec{c}))^i &= \epsilon^{ijk} a_j \epsilon^{klm} b_l c_m \\ &= \underbrace{\epsilon^{ijk} \epsilon^{klm}} a_j b_l c_m \end{aligned}$$

Think about it: for example, consider ϵ^{ij3} .
then for ϵ^{ij3} , is non-zero for $(i,j) = (1,2)$ and $(i,j) = (2,1)$,

$$\epsilon^{123} = -\epsilon^{213} = 1$$

So thinking along these lines we conclude

$$\epsilon^{ijk} \epsilon^{klm} = \boxed{\epsilon^{ijk} \epsilon^{lmk} = \delta^{il} \delta^{jm} - \delta^{im} \delta^{jl}}$$

So

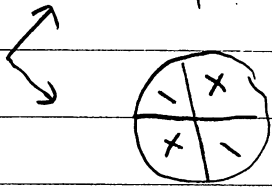
$$(\vec{a} \times (\vec{b} \times \vec{c}))^i = (\delta^{il} \delta^{jm} - \delta^{im} \delta^{jl}) a_j b_l c_m$$

$$= b^i (\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{b}) c^i, \text{ the "bac - abc" rule.}$$

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} (\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{b}) \vec{c}} \leftarrow \text{very important.}$$

Tensors

- Example: Want to describe the anisotropy of the charge distribution, and its orientation. Sort of described by two vectors. We will see that the right concept is the quadrupole tensor

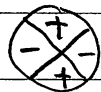


$$Q^{ij} = \int d^3x \rho(\vec{x}) \left(x^i x^j - \frac{1}{3} x^2 \delta^{ij} \right)$$

Rotations, rotate each arm of the tensor:

$$\underline{Q}^{ij} = R^i_{\ell} R^j_m Q^{\ell m}$$

↑ rotated tensor ↖ original tensor



Then, $\underline{Q} = Q^{ij} \vec{e}_i \vec{e}_j$, is a tensor and is unchanged under rotation of coordinates, since:

$$\vec{e}_i \vec{e}_j = \vec{e}_{\ell} \vec{e}_m (R^{-1})^{\ell}_i (R^{-1})^m_j$$

Derivative Operations:

$$\text{grad} = (\nabla \vec{S})_i = \partial_i S$$

$$\partial_i \equiv \frac{\partial}{\partial x^i}$$

$$\text{curl} = (\nabla \times \vec{V})^i = \epsilon^{ijk} \partial_j V_k$$

$$\text{div} = \nabla \cdot \vec{V} = \partial_i v^i = \partial_x V^x + \partial_y V^y + \partial_z V^z$$

$$\text{laplacian} \quad \nabla \cdot \nabla \vec{S} = \partial_i \partial^i$$

The $b(ac) - (ab)c$ rule plays an important role:

$$\nabla \times (\nabla \times \vec{C}) = \nabla (\nabla \cdot \vec{V}) - \nabla^2 \vec{C}$$

- Homework: use the $b(ac) - (ab)c$ rule to derive the wave equation