The Color Flavor Transformation

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in Jac Verbaarschot's Topics on Random Matrix Theory course

April 30, 1999

1 The Color Flavor Transformation

The Color Flavor Transformation was discovered by Martin Zirnbauer and relates an integral over random unitary $N \times N$ matrices to an integral over a pair of supermatrices Z and $\tilde{Z}$:

$$\int_{U(N)} dU\, exp\left(\bar{\psi}^i_{+a} U^{ij} \psi^j_{+a} + \bar{\psi}^j_{-b} \bar{U}^{ij} \psi^i_{-b}\right) = \int D\mu_N(Z, \tilde{Z})\, exp\left(\bar{\psi}^i_{+a} Z_{ab} \psi^i_{-b} + \bar{\psi}^j_{-b} \tilde{Z}_{ba} \psi^j_{+a}\right) \quad (1)$$

The transformation is exact for all N, including $N = 1$. Similar relations exist for an integral over $SO(2N)$ and $Sp(2N)$. The index a is a composite index $a = (\alpha, \sigma)$, where α runs from $1 \dots n_\pm$ according to $\pm a$ and σ is either B for bosonic or F for fermionic. This also fixes the type of the variables ψ. The bar denotes complex conjugation. We refer to the upper index of ψ as the color index and the lower index as the flavor index. We then could say that (1) integrates out the gluon fields, which carry color, and replaces them by an integral over color-singlet meson fields carrying flavor.

The new integration variables Z and $\tilde{Z}$ are supermatrices whose boson-fermion block decomposition is written as

$$Z = \begin{pmatrix} Z_{BB} & Z_{BF} \\ Z_{FB} & Z_{FF} \end{pmatrix} \qquad \tilde{Z} = \begin{pmatrix} \tilde{Z}_{BB} & \tilde{Z}_{BF} \\ \tilde{Z}_{FB} & \tilde{Z}_{FF} \end{pmatrix} .$$

Each block of Z ($\tilde{Z}$) is a rectangular matrix of dimension $n_+ \times n_-$ ($n_- \times n_+$). The integration measure is

$$D\mu_N(Z, \tilde{Z}) = D(Z, \tilde{Z})\, Sdet(1 - \tilde{Z}Z)^N ,$$

where $D(Z, \tilde{Z})$ denotes the flat Berezin measure on the space of supermatrices Z , $\tilde{Z}$. The domain of integration is defined by the conditions

$$\tilde{Z}_{BB} = Z^\dagger_{BB} \qquad \tilde{Z}_{FF} = -Z^\dagger_{FF}$$

and the requirement that all the eigenvalues of $\tilde{Z}_{BB} Z_{BB}$ are less than 1. The integration measure $D\mu_N$ is normalized to 1.

2 The geometric meaning of Z and $\tilde{Z}$

We argue that Z and $\tilde{Z}$ are coset representatives of Efetov's coset space G/H of unitary symmetry. (The latter is a Riemannian symmetric superspace of type $AIII \mid AIII$.)

Efetov's unitary σ-model space is G/H where $G = Gl(n|n)$, $n = n_+ + n_-$ and $H = Gl(n_+|n_+) \times Gl(n_-|n_-)$ which is the subspace of all matrices that commute with Σ_z, i.e.

$$\Sigma_z = \begin{pmatrix} 1_{n_+|n_+} & 0 \\ 0 & -1_{n_-|n_-} \end{pmatrix} \qquad H \ni h = \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix} .$$

The coset space G/H is a complex-analytic supermanifold and has complex dimension $d_{BB} = d_{FF} = 2n_+n_-$. We parameterize G/H by supermatrices $Q = g\Sigma_z g^{-1}$ and denote the G-invariant Berezin measure on G/H by DQ or Dg_H . When we integrate functions on G/H we take as domain of integration a Riemannian submanifold $M_{BB} \times M_{FF}$ of G/H, where

$$\begin{aligned} M_{BB} &= U(n_+, n_-)/U(n_+) \times U(n_-) \\ M_{FF} &= U(n_+ + n_-)/U(n_+) \times U(n_-) \end{aligned}$$

Because of convergence reason the integration domain has to be restricted to either a compact or a noncompact submanifold. We put the fermionic integration in the Berezinian. In this way one gets the G-invariant Berezin integral

$$\int_{M_{BB} \times M_{FF}} Dg_H \, f(gH) = \int_{M_{BB} \times M_{FF}} Dg_H \, f(g_0 gH) \qquad g_0 \in G .$$

For the proof given in the next section it is very essential that this invariant integral exists. (Note: We can undo the deformation of the integration domain by using Cauchy's theorem - assuming f is holomorphic.)

To compare with Z, $\tilde{Z}$ we parameterize G/H by decomposing $g \in G$ as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

and identify $h = \operatorname{diag}(A, D)$ and $Z = BD^{-1}$ and $\tilde{Z} = CA^{-1}$. The expression for $Q = g\Sigma_z g^{-1}$ in these coordinates is

$$Q = \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}^{-1} .$$

In these coordinates the G-invariant Berezin measure on G/H, DQ, turns out to be the flat one, $D(Z, \tilde{Z})$. This is also very important for the proof. Now we restrict ourselves to the Riemannian submanifold $M_{BB} \times M_{FF}$. For a unitary matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n_+ + n_-)$ we have $b^\dagger = -(d - ca^{-1}b)^{-1}ca^{-1}$ and $d^\dagger = (d - ca^{-1}b)^{-1}$. Therefore

$$Z_{FF}^\dagger = (bd^{-1})^\dagger = {d^{-1}}^\dagger b^\dagger = -ca^{-1} = -\tilde{Z}_{FF}$$

and for the $g \in U(n_+, n_-)$ the same except $b^\dagger = (d - ca^{-1}b)^{-1}ca^{-1}$ so

$$Z_{BB}^\dagger = \tilde{Z}_{BB} .$$

Moreover, $a^\dagger a - c^\dagger c = 1 \;\Rightarrow\; \tilde{Z}_{BB} Z_{BB} = ca^{-1}{a^{-1}}^\dagger c^\dagger = c(1 + c^\dagger c)^{-1} c^\dagger$ which requires all eigenvalues of $\tilde{Z}_{BB} Z_{BB}$ to be less than 1.

3 Proof of the Color Flavor Transformation

We show that the matrix elements of Z can be viewed as the parameters of a generalized coherent state obtained by the action of Efetov's symmetry group G on the ground state $|0\rangle$. An important property of coherent states we are going to use here is the resolution of the identity:

$$1_V = \int_K dk\, T_k|0\rangle\langle 0|T_k^\dagger$$

K is a compact Lie group and T is a unitary ($T_k^{-1} = T_k^\dagger$) irreducible action of K on a Fock space sector V with ground state $|0\rangle$. Irreducible means that all states of V can be reached by multiple action of T on the ground state. We call $T_k|0\rangle$ generalized coherent states. We normalize by $\langle 0|0\rangle = 1$. For quantum spin systems $K = SU(2)$ and V is a spin-S representation space of $SU(2)$. We now want to generalize this construction to $K = Gl(n|n)$ acting on G/H. For the Fock space of fermions and bosons we introduce another composite index $A = +a, -b$, i.e. $\{\psi^i_A\} = \{\psi^i_{+a}, \psi^i_{-b}\}$.

We introduce creation and annihilation operators with $c^i_A|0\rangle = 0$ and canonical supercommutation relations

$$[c^i_A, \bar{c}^j_B] = c^i_A\bar{c}^j_B - (-1)^{|A||B|}\bar{c}^j_B c^i_A = \delta_{AB}\delta^{ij},$$

and all other commutators equal to zero. Here, $|A| = 0$ if A = bosonic and $A = 1$ if A = fermionic. For later convenience we make the canonical transformation

$$c^i_{-b} \to \bar{c}^i_{-b} \qquad \bar{c}^i_{-b} \to (-1)^{|b|+1}c^i_{-b}\,.$$

The supercommutation relations stay the same. The annihilation operators of the ground state are now given by

$$\begin{aligned} c^i_{+a}|0\rangle &= 0 \qquad \langle 0|\bar{c}^i_{+a} = 0 \\ \bar{c}^i_{-b}|0\rangle &= 0 \qquad \langle 0|c^i_{-b} = 0\,. \end{aligned}$$

So we might say that the positive energy states are empty and the negative energy states are filled.

Next consider the Lie superalgebra $gl(nN, nN)$ of elements $E^{ij}_{AB} = \bar{c}^i_A c^j_B$:

$$\begin{aligned} [E^{ij}_{AB}, E^{kl}_{CD}] &= E^{ij}_{AB}E^{kl}_{CD} - (-1)^{(|A|+|B)(|C|+|D|)}E^{kl}_{CD}E^{ij}_{AB} \\ &= \delta_{BC}\delta^{jk}E^{il}_{AD} - (-1)^{(|A|+|B)(|C|+|D|)}\delta_{AD}\delta^{il}E^{kj}_{CB} \end{aligned}$$

The algebra $gl(nN, nN)$ has two commuting subalgebras namely

$$\begin{aligned} gl(N) &= \left\{\sum_A \bar{c}^i_A c^j_A\right\} \qquad \text{flavor singlet operators} \\ gl(n,n) &= \left\{\sum_i \bar{c}^i_A c^i_B\right\} \qquad \text{color singlet operators.} \end{aligned}$$

We define the flavor sector by demanding

$$\sum_A \bar{c}^i_A c^j_A|\text{flavor sector state}\rangle = 0 \quad \forall i,j\,.$$

The ground state lies in the flavor sector. We next show that the action of $gl(n,n)$ on the flavor sector is irreducible, therefore we will also call the flavor sector the color neutral sector. Let's first look at the charge neutral sector which contains the color neutral sector:

$$\sum_A \bar{c}^i_A c^i_A |\text{charge neutral sector state}\rangle = 0\,.$$

The charge neutral sector states consist of as many + particles as - particles which can be seen easily by writing out the sum over all flavors and transforming back to the original operators. From the Fock space representation of the charge neutral sector we see that this sector can be obtained by multiple action of $\bar{c}^i_{+a} c^j_{-b} \in gl(nN,nN)$ on the ground state. This means that $gl(nN,nN)$ acts irreducibly on the charge neutral sector. In order for the charge neutral sector state also be color neutral it has to be invariant under the transformations $R \in Gl(N)$ in color space, i.e.

$$\bar{c}^i_A \to R^{ij} \bar{c}^j_A \qquad\qquad c^i_A \to c^j_A (R^{-1})^{ji}\,.$$

For simplicity let's look at the charge neutral state $F^{ij}_{ab}\,\bar{c}^i_{+a} c^j_{-b}|0\rangle$. To be color neutral we get the condition $F^{ij} R^{ik} (R^{-1})^{lj} = F^{kl}$ which is solved by $F^{ij} = \delta^{ij}$. Therefore $gl(n,n)$ acts irreducibly on the color neutral sector.

For all elements of the corresponding Lie supergroup $Gl(n|n)$ we define an operator T_g on Fock space

$$T_g = \exp\left(\sum_{ABi} \bar{c}^i_A (\log(g))_{AB} c^i_B\right)\,.$$

Next we show that $T_g T_h = T_{gh}$. Consider

$$T_{\exp(X)} \bar{c}^i_A T^{-1}_{\exp(X)} = \exp\left(ad(\bar{c}^j_B X_{BC} c^j_C)\right) \bar{c}^i_A = \bar{c}^i_B (\exp(X))_{BA}$$

according to the Baker-Hausdorff formula (By $ad(X)$ we denote the operator that acts on A by $ad(X)A = [X,A]$.). For $\exp(X) = g$ we then find

$$T_g T_h \bar{c}^i_A T_h^{-1} T_g^{-1} = T_g (\bar{c}^i_B h_{BA}) T_g^{-1} = \bar{c}^i_C g_{CB} h_{BA} = \bar{c}^i_C (gh)_{CA} = T_{gh} \bar{c}^i_A T^{-1}_{gh}\,.$$

That means $T^{-1}_{gh} T_g T_h$ commutes with all $\bar{c}^i_A$ and similarly with all c^j_B, i.e. with all operators that act irreducibly on Fock space. According to Schur's lemma $T^{-1}_{gh} T_g T_h$ is a multiple of the identity. (Schur's lemma: Let A be a set of operators on a complex vector space V and suppose that V is irreducible under A. If C commutes with all the operators in A then C is a scalar multiple of the identity on V.) The constant of proportionality can be calculated easily to be 1 and therefore $T_g T_h = T_{gh}$.

The subgroup $H = Gl(n_+|n_+) \times Gl(n_-|n_-)$ stabilizes the vacuum:

$$\begin{aligned} T_{diag(A,D)}|0\rangle &= \exp\left(\bar{c}^i_{+a} (\log(A))_{ab} c^i_{+b} + \bar{c}^i_{-a} (\log(D))_{ab} c^i_{-b}\right)|0\rangle \\ &= |0\rangle \exp\left(N \sum_a (-1)^{|a|+1} (\log(D))_{aa}\right) = |0\rangle\ Sdet(D)^{-N}\,. \end{aligned}$$

Now consider the operator

$$P = \int_{M_{BB}\times M_{FF}} Dg_H\ T_g|0\rangle\langle 0|T_g^{-1}\,.$$

According to the above calculations the integrand is a function on the coset space G/H. It is easy to see that P commutes with all operators T_{g_0} due to the invariance of the integral under translations $g \to g_0 g$. Since the Lie superalgebra of $Gl(n|n)$ acts irreducibly on the flavor sector and since P commutes with all operators T_{g_0}, Schur's lemma again tells us that P is proportional to the identity of the flavor sector. Let's calculate the constant of proportionality. For this purpose we have to choose a gauge, i.e. pick a parameterization of g with parameters Z and $\tilde{Z}$ of the coset G/H. Following the parameterization of $G \ni g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in the previous section we choose

$$\begin{aligned} g(Z,\tilde{Z}) &= \begin{pmatrix} (1-Z\tilde{Z})^{-1/2} & Z(1-\tilde{Z}Z)^{-1/2} \\ \tilde{Z}(1-Z\tilde{Z})^{-1/2} & (1-\tilde{Z}Z)^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1-Z\tilde{Z})^{1/2} & 0 \\ 0 & (1-\tilde{Z}Z)^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tilde{Z} & 1 \end{pmatrix} . \end{aligned}$$

In fact $Z = BD^{-1}$ and $\tilde{Z} = CA^{-1}$. With this parameterization we get

$$\begin{aligned} T_{g(Z,\tilde{Z})}|0\rangle &= \exp\left(\bar{c}^i_{+a} Z_{ab} c^i_{-b}\right) \exp\left(\frac{1}{2}\bar{c}^i_{+a} \log(1-Z\tilde{Z})_{ab} c^i_{+b} - \frac{1}{2}\bar{c}^i_{-a} \log(1-\tilde{Z}Z)_{ab} c^i_{-b}\right) \\ &\quad \times \exp\left(\bar{c}^i_{-a} \tilde{Z}_{ab} c^i_{+b}\right) |0\rangle \\ &= \exp\left(\bar{c}^i_{+a} Z_{ab} c^i_{-b}\right) |0\rangle \; Sdet(1-\tilde{Z}Z)^{N/2} =: |Z\rangle \\ \langle 0| T^{-1}_{g(Z,\tilde{Z})} &= T_{g(-Z,-\tilde{Z})} = Sdet(1-\tilde{Z}Z)^{N/2} \, \langle 0| \exp\left(-\bar{c}^i_{-a} \tilde{Z}_{ab} c^i_{+b}\right) =: \langle Z| \, . \end{aligned}$$

For the invariant measure in these coordinates we have $D(Z,\tilde{Z})$, therefore

$$P = \int D(Z,\tilde{Z}) \; |Z\rangle\langle Z|$$

and for the vacuum expectation value of P we get 1 by our choice of normalization of $D\mu_N(Z,\tilde{Z})$. So P is the identity on the flavor sector and P annihilates all states outside the flavor sector since $\langle Z|$ vanishes on states that are not color singlets. This means:

P projects Fock space onto the flavor sector.

Finally we need coherent states of bosons and fermions:

$$\exp\left(\bar{c}^i_{+a}\psi^i_{+a} + \bar{\psi}^i_{-b} c^i_{-b}\right) |0\rangle \, .$$

These coherent states of bosons and fermions span the entire Fock space. We are going to project them on the flavor or color neutral sector by making a unitary rotation in color space $c^i_A \to c^j_A U^{ji}, \bar{c}^i_A \to \bar{c}^j_A \bar{U}^{ji}$ and averaging over all such rotations:

$$P \exp\left(\bar{c}^i_{+a}\psi^i_{+a} + \bar{\psi}^i_{-b} c^i_{-b}\right) |0\rangle = \int_{U(N)} dU \; \exp\left(\bar{U}^{ji}\bar{c}^j_{+a}\psi^i_{+a} + \bar{\psi}^i_{-b} c^j_{-b} U^{ji}\right) |0\rangle \, .$$

Now we can put the pieces together for the proof of the Color Flavor Transformation (1): $\int D\mu_N(Z,\tilde{Z}) \, exp\left(\bar{\psi}^i_{+a} Z_{ab} \psi^i_{-b} + \bar{\psi}^j_{-b} \tilde{Z}_{ba} \psi^j_{+a}\right)$

$$= \int D\mu_N(Z,\tilde{Z}) \, \langle 0| \exp\left(\bar{\psi}^i_{+a} c^i_{+a} - \bar{c}^i_{-b}\psi^i_{-b}\right) \exp\left(\bar{c}^i_{+a} Z_{ab} c^i_{-b}\right) |0\rangle$$

$$
\begin{aligned}
& \times \langle 0| \exp\left(-\bar{c}^i_{-b}\tilde{Z}_{ba}c^i_{+a}\right) \exp\left(\bar{c}^i_{+a}\psi^i_{+a} + \bar{\psi}^i_{-b}c^i_{-b}\right)|0\rangle \\
= \; & \langle 0| \exp\left(\bar{\psi}^i_{+a}c^i_{+a} - \bar{c}^i_{-b}\psi^i_{-b}\right)\left(\int D(Z,\tilde{Z})\;|Z\rangle\langle Z|\right) \\
& \times \exp\left(\bar{c}^i_{+a}\psi^i_{+a} + \bar{\psi}^i_{-b}c^i_{-b}\right)|0\rangle \\
= \; & \langle 0| \exp\left(\bar{\psi}^i_{+a}c^i_{+a} - \bar{c}^i_{-b}\psi^i_{-b}\right) P \exp\left(\bar{c}^i_{+a}\psi^i_{+a} + \bar{\psi}^i_{-b}c^i_{-b}\right)|0\rangle \\
= \; & \int_{U(N)} dU\;\langle 0| \exp\left(\bar{\psi}^i_{+a}c^i_{+a} - \bar{c}^i_{-b}\psi^i_{-b}\right) \exp\left(\bar{c}^i_{+a}U^{ij}\psi^j_{+a} + \bar{\psi}^i_{-b}\bar{U}^{ji}c^j_{-b}\right)|0\rangle \\
= \; & \int_{U(N)} dU\; exp\left(\bar{\psi}^i_{+a}U^{ij}\psi^j_{+a} + \bar{\psi}^i_{-b}\bar{U}^{ji}\psi^j_{-b}\right)\;.
\end{aligned}
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4 Remarks on some applications of the Color Flavor Transformation

By using a supersymmetric variant of the Itzykson-Zuber formula one can use the Color Flavor Transformation to compute the n-level correlation functions of Dyson's circular unitary ensemble for any n in the large N limit. Other applications are random network models, quantum chaotic maps and QCD in the large N limit (Witten argued that large N QCD is equivalent to a weakly coupled theory of mesons) or lattice gauge theory, where instead of supermatrices Z and $\tilde{Z}$ we only have their FF blocks. (Unfortunately QCD has an $SU(N)$ gauge group and the exact gluon-meson transformation as it is discussed here is for gauge groups $U(N)$.)

References

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