Spectral correlations of the QCD Dirac operator at finite temperature

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1 Introduction

We want to calculate spectral correlation functions for a simplified random matrix model for Euclidean QCD at finite temperature by using the supersymmetric method. The calculation will focus on the quenched case, without dynamical quarks. The difference with calculations at $T = 0$ lies in the fact that unitary invariance of the model is broken by the terms arising from the temperature dependence. Remarkably, it turns out that nonetheless the correlation functions in the microscopic limit are universal, i. e. they have the same functional form as in the case $T = 0$, and the temperature dependence enters only through a scaling factor for the eigenvalue variables. The result hence proves that the correlation functions of the chiral Gaussian unitary ensemble in the microscopic limit are universal with respect to deformation of the distribution through the introduction of a temperature dependence.

This calculation was done by A.D. Jackson, M.K. Şener and J.J.M. Verbaarschot in 1997 [1], and simultaneously by T. Guhr and T. Wettig [2]. The calculation uses the graded eigenvalue method developed by T. Guhr [4]. We will largely follow the treatment in [1].

2 The model

The QCD partition function for N_f massless flavors is given by

$$\begin{aligned} Z_{QCD} &= \int d[A]d[\psi]d[\overline{\psi}] \exp\left(- \sum_{f=1}^{N_f} \int d^4x\, \overline{\psi}^f \,(i\gamma_\mu D_\mu)\, \psi^f \right) \exp(-S_{YM}) \\ &= \int d[A] \det{}^{N_f}\, (i\gamma_\mu D_\mu) \exp(-S_{YM}) \\ &= \left\langle \det{}^{N_f}\, (i\gamma_\mu D_\mu) \right\rangle_{S_{YM}} \end{aligned} \tag{1}$$

where $D_\mu = \partial_\mu + A_\mu$ is the covariant derivative with the gluon gauge field A_μ, and S_{YM} is the Yang-Mills-action of the gluon field. For finite temperature, the fermion fields are expanded in a basis

$$\psi(\mathbf{x}, \tau) = \sum_{k=1}^{N} \sum_{l=-n+1}^{n} \phi_{kl}(\mathbf{x}) \exp(i(2l-1)\pi T \tau) \tag{2}$$

where one has antiperiodic boundary conditions for the fermion fields on the Euclidean time axis, and the length of this axis is given by $1/T$. Only the n first Matsubara-frequencies are retained in the expansion. We identify N as the volume of space.

One now splits the Dirac operator in a piece $\gamma_0 \partial_\tau$ and a remainder. Observing the chiral structure of the Dirac operator the partition function in a chiral basis can be expressed as

$$Z_{QCD} \;=\; \left\langle \det{}^{N_f} \left[\begin{pmatrix} 0 & iC^\dagger \\ iC & 0 \end{pmatrix} + \begin{pmatrix} 0 & i\Theta \otimes 1_N \\ i\Theta \otimes 1_N & 0 \end{pmatrix} \right] \right\rangle_{S_{YM}} \tag{3}$$

The gauge field dependence is entirely in the complex $2nN \times 2nN$-matrix C. The temperature dependence is contained in the matrix $\Theta = \mathrm{diag}(-(2n-1)\pi T, \ldots, -\pi T, \pi T, \ldots, (2n-1)\pi T)$.

From this model, we get to the random matrix model by replacing the gauge-field dependent matrix elements of C by random variables with a Gaussian distribution, i.e. instead of averaging over the actual gauge field configuration, we average over a random distribution. As a further simplification we only retain the lowest Matsubara frequencies $\pm\pi T$. It presents no difficulties to include the full matrix Θ. In our simplified case, we can perform a unitary transformation on the truncated matrix Θ (and C), so that we are left with only positive Matsubara frequencies, and reduce the size of the matrix model by half (redefine N). It is then sufficient to look at

$$iD = i \begin{bmatrix} 0 & C + \pi T \mathbf{1}_N \\ C^\dagger + \pi T \mathbf{1}_N & 0 \end{bmatrix} \tag{4}$$

The random matrix model then finally becomes

$$Z_{RMT} = \mathcal{N} \int d[C] \exp(-N\Sigma^2 \mathrm{Tr} C^\dagger C) \det{}^{N_f}(D) \tag{5}$$

where the parameter Σ sets the scale of the model and is identified with the value of the chiral condensate at $T = 0$, and $d[C]$ denotes the Haar measure. We will restrict our calculation to the so-called "quenched" case for which $N_f = 0$. The average $\langle \ldots \rangle$ denotes an average with respect to the probability distribution given by the integral above.

For this model, the unitary invariance of the partition function is broken. We have chosen a particular basis for the representation of the Dirac operator, in which the temperature dependent part becomes diagonal. Unitary transformations on the matrix C correspond to a change of basis, so that for a general transformation the matrix Θ would no longer be diagonal, which would change our partition function. This is why the model is no longer invariant under unitary transformations, although the probability distribution for C of course still is. This complicates the calculation considerably because it is no no longer possible to use the "eigenvalues" of C as new integration variables.

2.1 Correlation functions

We are interested in calculating the quantities (x stands for $x\mathbf{1}_N$)

$$\hat{R}_k(x_1, \ldots, x_k) \;=\; \left\langle \prod_{p=1}^{k} \frac{1}{\pi} \mathrm{Tr} \left(\frac{-1}{x_p + i\epsilon_p - D} \right) \right\rangle \tag{6}$$

from which we can find the actual k-point eigenvalue correlation functions by taking the imaginary parts of the individual traces (and neglecting all parts containing δ-functions of two eigenvalues). In the following, x is understood to contain the infinitesimal imaginary part $+i\epsilon$, the above expressions are valid in the limit $\epsilon \to 0$.

The quantities $\hat{R}_k$ can be written in terms of a generating functional as

$$\hat{R}_k(x_1, \ldots, x_k) \;=\; \frac{1}{(2\pi)^k} \prod_{p=1}^{k} \frac{\partial}{\partial j_p} Z_k(j_1, \ldots, j_k)\Big|_{j_p=0} \tag{7}$$

where the functional is given by

$$Z_k(j_1, \ldots, j_k) \;=\; \mathcal{N} \int d[C] \exp(-N\Sigma^2 \mathrm{Tr} C^\dagger C) \det{}^{N_f} D \prod_{p=1}^{k} \frac{\det(D - x_p + j_p)}{\det(D - x_p - j_p)} \tag{8}$$

As mentioned before, we will restrict ourselves to the quenched case for which the fermion-determinant is absent.

3 Outline of calculation

The calculation consists of the following main steps. The quotient of determinants in the generating functional will be expressed in terms of a superdeterminant of a supermatrix. This superdeterminant in turn can be written as a Gaussian integral over supervectors containing commuting and anticommuting variables. The average over the random variables can then be performed by completing a square. This leads to terms of 4th order in the superfields, over which an integration cannot be performed. To decouple these fields again, a Hubbard-Stratonovich-transformation is performed, i.e. an additional supermatrix integration is introduced. After decoupling, one would like to perform the integration, but to do that, one needs to interchange the superfield integration with the one over the the decoupling matrix field, which can only be done with great care. In the end, one can do the superfield integration and one obtains the superdeterminant of a supermatrix of size $(2k|2k) \times (2k|2k)$ for the k-point functional, which contains N only as a parameter and is easier to handle than the superdeterminant of a $(2kN|2kN)$-supermatrix we started out with. The only integration left is the one over the decoupling matrix. To do this integration, one pseudo-diagonalizes this decoupling-matrix and rewrites it in terms of angular and polar coordinates. The angular integrations, which are group integrations over the diagonalizing groups, can then be performed as the supersymmetric generalization of an Itzykson-Zuber integral. Finally, after differentiation with respect to the source terms in the generating functional, this leads to the correlation functions in terms of a determinantal structure over kernels which still contain integrations over the polar coordinates. In the microscopic limit, these can be evaluated by saddle point integrations.

4 Calculation

4.1 Introduction of supervectors

We now introduce the $(2kN|2kN) \times (2kN|2kN)$-supermatrices

$$\mathbf{C} \;=\; \mathrm{diag}(C, \ldots, C; C, \ldots, C)$$

$$
\begin{aligned}
\mathbf{x} &= \mathrm{diag}(x_1\mathbf{1}_N, \ldots, x_k\mathbf{1}_N; x_1\mathbf{1}_N, \ldots, x_k\mathbf{1}_N) \\
\mathbf{j} &= \mathrm{diag}(-j_1\mathbf{1}_N, \ldots, -j_k\mathbf{1}_N; j_1\mathbf{1}_N, \ldots, j_k\mathbf{1}_N)
\end{aligned} \tag{9}
$$

with k boson-boson-blocks of size $N \times N$ and k fermion-fermion-blocks of same size on the diagonal. We then construct a matrix with a supermatrix-block-structure from this. For a matrix of this structure, the determinant factorizes in determinants of sub-blocks, so that we have

$$
\begin{aligned}
&\mathrm{sdet}\begin{bmatrix} -\mathbf{x}+\mathbf{j} & \mathbf{C}^\dagger + \pi T\mathbf{1}_{2kN} \\ \mathbf{C}+\pi T\mathbf{1}_{2kN} & -\mathbf{x}+\mathbf{j} \end{bmatrix} \\
&= \prod_{p=1}^{k} \mathrm{sdet}\left[\begin{array}{cc|cc} -x_p - j_p & 0 & C^\dagger + \pi T & 0 \\ 0 & -x_p + j_p & 0 & C^\dagger + \pi T \\ \hline C + \pi T & 0 & -x_p - j_p & 0 \\ 0 & C+\pi T & 0 & -x_p + j_p \end{array}\right] \\
&= \prod_{p=1}^{k} \frac{\det\begin{bmatrix} -x_p - j_p & C^\dagger + \pi T \\ C + \pi T & -x_p - j_p \end{bmatrix}}{\det\begin{bmatrix} -x_p + j_p & C^\dagger + \pi T \\ C + \pi T & -x_p + j_p \end{bmatrix}}
\end{aligned} \tag{10}
$$

Now we introduce 2 supervectors $\psi_A = (\phi_A, \chi_A)^T$ and an analogous ψ_B, with bosonic (commuting) components ϕ_A and fermionic (anticommuting) components χ_A, of size kN each, so that the ψ are $(kN|kN)$ supervectors. Then we can express the superdeterminant above as an integration over these fields:

$$
\begin{aligned}
&\mathrm{sdet}^{-1}\begin{bmatrix} -\mathbf{x}+\mathbf{j} & \mathbf{C}^\dagger + \pi T\mathbf{1}_{2kN} \\ \mathbf{C}+\pi T\mathbf{1}_{2kN} & -\mathbf{x}+\mathbf{j} \end{bmatrix} \\
&= \int d[\psi_A]d[\psi_B] \exp\left(-i \begin{pmatrix} \psi_A^\dagger & \psi_B^\dagger \end{pmatrix} \begin{bmatrix} -\mathbf{x}+\mathbf{j} & \mathbf{C}^\dagger + \pi T\mathbf{1}_{2kN} \\ \mathbf{C}+\pi T\mathbf{1}_{2kN} & -\mathbf{x}+\mathbf{j} \end{bmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}\right)
\end{aligned} \tag{11}
$$

Inserting this into the generating functional Z_k, we can evaluate the disorder average.

4.2 Disorder average

To calculate the integration over the matrix C, we rewrite the terms containing the products of the supervectors and the matrix as (e.g.)

$$
\begin{aligned}
\psi_A^\dagger \mathbf{C}^\dagger \psi_B &= \sum_{p=1}^{k} \phi_{Ap}^\dagger C^\dagger \phi_{Bp} + \chi_{Ap}^\dagger C^\dagger \chi_{Bp} \\
&= \sum_{p=1}^{k} \sum_{i,j=1}^{N} C_{ij}^\dagger \phi_{Apj}^* \phi_{Bpi} - C_{ij}^\dagger \chi_{Apj}^* \chi_{Bpi} \\
&= \sum_{p=1}^{k} \mathrm{Tr}(C^\dagger E_p^\dagger)
\end{aligned} \tag{12}
$$

where the matrix $E^\dagger$ is given by

$$
E_p^\dagger = \phi_{Bp} \otimes \phi_{Ap}^\dagger - \chi_{Bp} \otimes \chi_{Ap}^\dagger. \tag{13}
$$

In the same way we have

$$\psi_B^\dagger \mathbf{C} \psi_A = \sum_{p=1}^{k} \mathrm{Tr}(CE_p). \tag{14}$$

These are the only terms containing a C-dependence, so only these will be affected by the average. Now we use the identity

$$\int d[C] \exp(-N\Sigma^2 \mathrm{Tr}(C^\dagger C) - i\mathrm{Tr}(C^\dagger E^\dagger + CE)) \;=\; \mathcal{N}^{-1} \exp\left(-\frac{1}{n\Sigma^2}\mathrm{Tr}(E^\dagger E)\right) \tag{15}$$

which can be obtained by completing a square. At this point, our partition function looks like

$$\begin{aligned} Z \;=\; & \mathcal{N} \int d[C] \exp(-N\Sigma^2 \mathrm{Tr} C^\dagger C) \\ & \times \int d[\psi] \exp\left[-i \sum_p \mathrm{Tr}(CE_p + C^\dagger E_p^\dagger)\right] \\ & \times \exp\left[-i\left(\psi_A^\dagger(-\mathbf{x}+\mathbf{j})\psi_A + \psi_B^\dagger(-\mathbf{x}+\mathbf{j})\psi_B\right)\right] \\ & \times \exp(-i\pi T(\psi_A^\dagger \cdot \psi_B + \psi_B^\dagger \cdot \psi_A)) \end{aligned} \tag{16}$$

In order to perform the average, we have to interchange the integrations over the ψ and over C. This can be done, since the fermionic integrations are convergent anyway, and the $i\epsilon$-parts in the x variables insure convergence for the bosonic integrations. After interchanging and performing the integrations, we obtain the result

$$\begin{aligned} Z \;=\; & \int d[\psi] \exp\left[\frac{1}{n\Sigma^2}\mathrm{Tr}(E^\dagger E) + i\left(\psi_A^\dagger(-\mathbf{x}+\mathbf{j})\psi_A + \psi_B^\dagger(-\mathbf{x}+\mathbf{j})\psi_B\right)\right] \\ & \times \exp\left[-i\pi T(\psi_A^\dagger \cdot \psi_B + \psi_B^\dagger \cdot \psi_A)\right] \end{aligned} \tag{17}$$

4.3 Hubbard-Stratonovich-transformation

In doing the disorder average, we have arrived at terms of the form $\mathrm{Tr}(E^\dagger E) \sim \psi^4$, and unfortunately we cannot directly perform the ψ-integrations over these terms. Therefore, we have to decouple them, by using a Hubbard-Stratonovich-transformation, i.e. we again complete a square. First, we observe that there is a way to rewrite the trace over the matrices E as a supertrace over two newly introduced supermatrices. We can write

$$\begin{aligned} \mathrm{Tr}(E^\dagger E) \;&=\; \mathrm{Str}(AB) \\ &=\; \mathrm{Str}\left[\begin{bmatrix} (\phi_A^\dagger \cdot \phi_A) & (\chi_A^\dagger \cdot \phi_A) \\ (\phi_A^\dagger \cdot \chi_A) & (\chi_A^\dagger \cdot \chi_A) \end{bmatrix} \begin{bmatrix} (\phi_B^\dagger \cdot \phi_B) & (\chi_B^\dagger \cdot \phi_B) \\ (\phi_B^\dagger \cdot \chi_B) & (\chi_B^\dagger \cdot \chi_B) \end{bmatrix}\right] \end{aligned} \tag{18}$$

The Hubbard-Stratonovich-identity is then, with σ a non-hermitian, complex supermatrix of dimension $(k|k)$,

$$\exp\left(-\frac{1}{N\Sigma^2}\mathrm{Str}(AB)\right) \;=\; \int d[\sigma] \exp(-N\Sigma^2 \mathrm{Str}(\sigma^\dagger\sigma) - i\mathrm{Str}(\sigma^\dagger A + \sigma B)) \tag{19}$$

This decouples the 4th-order terms in ψ and solves this part of the problem. Further, we rewrite the supertraces linear in A and B in the form

$$\begin{aligned} \mathrm{Str}(\sigma^\dagger A) &= \psi_A^\dagger(\tilde{\sigma} \otimes \mathbf{1}_N)\psi_A \\ \mathrm{Str}(\sigma B) &= \psi_B^\dagger(\sigma \otimes \mathbf{1}_N)\psi_B \end{aligned} \tag{20}$$

with $\tilde{\sigma} = \sigma^\dagger K$, $K = \mathrm{diag}(\mathbf{1}_k, -\mathbf{1}_k)$. Also, we shift the integration variables σ by the matrices in the other terms that are diagonal in the indices A, B:

$$\begin{aligned} \sigma &\to \sigma + (\mathbf{x} - \mathbf{j}) \\ \tilde{\sigma} &\to \tilde{\sigma} + (\mathbf{x} - \mathbf{j}) \end{aligned} \tag{21}$$

Here, we redefine $\mathbf{x}, \mathbf{j}$ as diagonal supermatrices of size $(k|k)$, i.e. they no longer have blocks $x_p\mathbf{1}_N, j_p\mathbf{1}_N$ on the diagonal, but only simple entries x_p, j_p. The infinitesimal imaginary parts $i\epsilon$ remain with the σ matrices that are sandwiched between the ψ-fields. Putting these results together, we arrive at

$$\begin{aligned} &Z_k(j_1, \ldots, j_k) = \\ &= \int d[\psi] \int d[\sigma] \exp[-N\Sigma^2 \mathrm{Str}((\tilde{\sigma} + \mathbf{x} - \mathbf{j})(\sigma + \mathbf{x} - \mathbf{j}))] \\ &\quad \times \exp\left(-i\psi^\dagger \left[\begin{array}{cc} \tilde{\sigma}_- \otimes \mathbf{1}_N & \pi T \mathbf{1}_{2k} \otimes \mathbf{1}_N \\ \pi T \mathbf{1}_{2k} \otimes \mathbf{1}_N & \sigma_- \otimes \mathbf{1}_N \end{array} \right] \psi \right) \end{aligned} \tag{22}$$

where we have finally reduced our partition function in such a way that it only contains a supermatrix of size $(k|k)$, and N does no longer appear in the size of a matrix but only in a summation over components of the supervectors:

$$\psi^\dagger \left[\begin{array}{cc} \tilde{\sigma}_- \otimes \mathbf{1}_N & \pi T \mathbf{1}_{2k} \otimes \mathbf{1}_N \\ \pi T \mathbf{1}_{2k} \otimes \mathbf{1}_N & \sigma_- \otimes \mathbf{1}_N \end{array} \right] \psi = \sum_{i=1}^N \psi_i^\dagger \left[\begin{array}{cc} \tilde{\sigma}_- & \pi T \otimes \mathbf{1}_{2k} \\ \pi T \otimes \mathbf{1}_{2k} & \sigma_- \end{array} \right] \psi_i \tag{23}$$

We are now able to do the integrations over the superfields, except for one last obstacle. We need to interchange the integrations over σ and over ψ. Initially, it seems that this is impossible. The matrix σ is arbitrary complex (our $i\epsilon$-factors no longer rescue us), so it is no longer assured that the integrations over the bosonic parts of the superfields are uniformly convergent, which is the condition for the interchangeability of the integrations. In the case of the one-point-function it is possible to find a parameterization for the matrix σ which renders the integrations convergent [3]. Unfortunately, this method cannot be easily generalized to higher order correlation functions. But it turns out that a naive interchange of the integrations in case of the one-point function still leads to the right result for its imaginary part. It was therefore conjectured that it is also possible to obtain the imaginary parts of the higher-order correlation functions by a formal evaluation of the integral [1, 2]. Following this conjecture, we interchange the integrations and evaluate

$$\begin{aligned} &Z_k(j_1, \ldots, j_k) = \\ &= \int d[\sigma] \exp[-N\Sigma^2 \mathrm{Str}(\tilde{\sigma} + \mathbf{x} - \mathbf{j})(\sigma + \mathbf{x} - \mathbf{j})] \mathrm{sdet}^{-N} \left[\begin{array}{cc} \tilde{\sigma}_- & \pi T \otimes \mathbf{1}_{2k} \\ \pi T \otimes \mathbf{1}_{2k} & \sigma_- \end{array} \right] \end{aligned} \tag{24}$$

4.4 Calculation of integrations over the decoupling field σ

Now the only integration left to do is the one over the decoupling matrix σ. This integration is done in two steps: the matrix σ is pseudo-diagonalized with two superunitary matrices, and the integrations are performed separately over the diagonalizing variables and the eigenvalues of the matrix. The matrix σ can be decomposed as $\sigma = U^\dagger SV$, where $S = \mathrm{diag}(s_1, \ldots, s_k; i\xi_1, \ldots, i\xi_k)$, where s_p,ξ_p are real and positive, and $U^\dagger U = \mathbf{1}$,$V^\dagger KV = K$. U parameterizes $U(k|k)$ and V parameterizes $U(k|k)/U^{k+k}(1)$. Notice that V is used to absorb the phases of the eigenvalues so that they become real positive ones. In terms of these new variables the integration measure is given by

$$d[\sigma] = J(S)d[S^2]d\mu(U)d\mu(V) \tag{25}$$

The last two terms denote the invariant group measures over the superunitary groups and $J(S)$ is the Jacobian of the transformation. It is given by the Berezinian

$$\begin{aligned} J(S) &= B_k^2(S^S) \\ B_k(S^2) &= \det\left[\frac{1}{s_p^2 - (i\xi_q)^2}\right]_{p,q=1,\ldots,k} \end{aligned} \tag{26}$$

$d[S^2]$ is given by $d[S^2] = \prod_{p=1}^k d(s_p^2)d(\xi_p^2)$. For the structure of the correlation functions which we would like to obtain, it is crucial that the Berezinian has a determinantal structure. Only then is it possible to express the correlation functions in terms of determinants of a kernel. If the supermatrices do not have fermionic and bosonic blocks of same size, the Berezinian cannot be written in determinantal form. This is the most serious obstacle which the calculation in the unquenched case (i.e. with an additional determinant in the partition function) faces, above those already present for the quenched calculation.

4.5 Itzykson-Zuber integration

The group integrations to be done are a supersymmetric generalization of an Itzykson-Zuber type integral. As for Itzykson-Zuber integrals, they can be evaluated by means of a method describing them as the solutions of a diffusion-type equation. The integral we need here was worked out in [6]. Setting $\rho = \mathrm{diag}(x_1 + j_1, \ldots, x_k + j_k; x_1 - j_1, \ldots, x_1 - j_1)$ and identifying this with $\rho = \mathrm{diag}(r_1, \ldots, r_k; i\eta_1, \ldots, i\eta_k)$, it is given by

$$\begin{aligned} &\int d\mu(U) \int d\mu(V) \exp(\mathrm{Str}(\tilde{\sigma} - \rho)(\sigma - \rho) = \\ &= \frac{1}{(k!)^2} \frac{\det[\gamma(s_p, r_q)]_{p,q=1,\ldots,k} \det[\gamma(\xi_p, \eta_q)]_{p,q=1,\ldots,k}}{B(S^2)B(R^2)} \end{aligned} \tag{27}$$

where the γ denote the functions (with $I_\nu(z)$ modified Bessel functions)

$$\gamma(s,r) = \exp(-s^2 - r^2)I_0(2sr) \tag{28}$$

We have absorbed the prefactor $N\Sigma^2$ in the variables s, r, ξ, η, so that we have to keep in mind that e.g $sr \propto N\Sigma^2$ to later restore this dependence.

4.6 Kernel and determinant expression for the correlation functions

Now we need to expand the last superdeterminant present. Because of the block structure with diagonal matrices, it again factorizes into blocks of supermatrices and one finds

$$\begin{aligned}
\mathrm{sdet}^{-N}\begin{bmatrix} S & \pi T\mathbf{1}_k \\ \pi T\mathbf{1}_k & S \end{bmatrix} &= \left(\prod_{p=1}^{k}\mathrm{sdet}\left[\begin{array}{cc|cc} s_p & 0 & \pi T & 0 \\ 0 & i\xi_p & 0 & \pi T \\ \hline \pi T & 0 & s_p & 0 \\ 0 & \pi T & 0 & i\xi_p \end{array}\right]\right)^{-N} \\
&= \left[\prod_{p=1}^{k}\frac{\det\begin{pmatrix} s_p & \pi T \\ \pi T & s_p \end{pmatrix}}{\det\begin{pmatrix} i\xi_p & \pi T \\ \pi T & i\xi_p \end{pmatrix}}\right]^{-N} = \prod_{p=1}^{k}\left(\frac{-(\pi T)^2-\xi_p^2}{-(\pi T)^2+s_p^2}\right)^{N} \qquad (29)
\end{aligned}$$

Apart from normalization factors, the generating functional is at this point

$$\begin{aligned}
Z(j_1,\ldots,j_k) &\propto \frac{1}{B(R^2)}\int_0^\infty\prod_{p=1}^{k}4s_p\xi_p ds_p d\xi_p\left(\frac{\xi_p^2+(\pi T)^2}{s_p^2-(\pi T)^2}\right)^N \\
&\times\frac{1}{(k!)^2}\det[\gamma(s_p,r_q)]_{p,q=1,\ldots,k}\det\left[\frac{1}{s_p^2+\xi_q^2}\right]_{p,q=1,\ldots,k}\det[\gamma(\xi_p,\eta_q)]_{p,q=1,\ldots,k} \qquad (30)
\end{aligned}$$

The determinants in the integrations can be joined together, which one can see by writing out the determinants explicitly and combining the functions with the same integration variables. The following identity holds in general:

$$\frac{1}{k!}\int\prod_{p=1}^{k}dy_p\det[f(x_p,y_q)]_{p,q=1,\ldots,k}\det[g(y_p,z_q)]_{p,q=1,\ldots,k} = \det\left[\int dy f(x_p,y)g(y,z_q)\right]_{p,q=1,\ldots,k} \qquad (31)$$

Applying this identity twice to the above expression, and distributing the additional products over integration variables appropriately onto the determinants, the functional can be written in terms of a single determinant over double integrations.

To obtain the correlation functions, one needs to take the derivative with respect to the source variables $\{j_p\}$, and to evaluate this at $\{j_p=0\}$. Since e.g. $r_p=x_p+j_p$, we have a j-dependence in the kernels in the determinant expressions as well as in the prefactor $1/B(R^2)$. Fortunately, remembering $r_p=x_p+j_p$, $i\eta_p=x_p-j_p$, we see that at $\{j_p=0\}$ the inverse Berezinian $1/B(R^2)$ contains terms which are zero in each product and hence vanishes. This means that, taking the derivative according to the product rule, the contribution which could eventually contain derivatives of the determinant expressions will vanish on setting the $j_p=0$, and only the term containing the derivative of the Berezinian as its prefactor gives a contribution. The derivative of $1/B(R^2)$ with respect to $j_1,\ldots,j_k$ at $\{j_p=0\}$ turns out to be simply $4\prod_{p=1}^{k}x_p$. This can be distributed symmetrically as factors $\sqrt{x_px_q}$ onto the terms in the determinant. Restoring the factors $N\Sigma^2$ in the variables (where necessary) and defining $t=\pi T\Sigma$, we can finally write the correlation functions

$$\hat{R}(x_1,\ldots,x_k)=\det[\Sigma\hat{K}_N(x_p\Sigma,x_q\Sigma)]_{p,q=1,\ldots,k} \qquad (32)$$

where we have used the kernel defined by

$$\hat{K}_N(x_p\Sigma, x_q\Sigma) = -\frac{8N^2\Sigma}{\pi}\sqrt{x_p x_q}\int_0^\infty d\xi \int_0^\infty ds \frac{s\xi}{s^2+\xi^2}\left(\frac{t^2+\xi_2}{t^2-(s-i\epsilon)^2}\right)^N$$
$$\times \exp(-N[s^2+\xi^2+\frac{1}{2}\Sigma^2(x_p^2-x_q^2)])\ I_0(2Ns\Sigma x_p)\ I_0(i2N\xi x_q) \qquad (33)$$

This is the main result of the calculation. The integrals still present can be evaluated in the special limits one is interested in. We have explicitly written out the $-i\epsilon$ term again. To obtain the actual correlation functions it is necessary to take the imaginary parts of the individual terms.

5 Microscopic limit

The microscopic limit is the limit in which $N \to \infty$, and the spectrum is simultaneously rescaled by a factor N, i.e. the area close to the origin of the spectrum will be blown up, and the eigenvalues close to the origin will be resolved. It is of particular interest, since the eigenvalue density near the origin allows a statement about the value of the chiral condensate.

5.1 $T = 0$

The double-integrations can be separated by a Feynman-parameterization, and the integrals over s and ξ can be evaluated exactly. Setting $\tilde{x} = Nx$, one finds in the microscopic limit

$$\begin{aligned} K_S(\tilde{x}, \tilde{x}) &= \lim_{N\to\infty} K_N(\frac{\tilde{x}}{N}, \frac{\tilde{y}}{N}) \\ &= \frac{\sqrt{\tilde{x}\tilde{y}}}{\tilde{x}^2-\tilde{y}^2}(\tilde{x}J_0(2\tilde{y})J_1(2\tilde{x}) - \tilde{y}J_0(2\tilde{x})J_1(2\tilde{y})) \end{aligned} \qquad (34)$$

i.e. the calculation reproduces the well-known Bessel-kernel, as indeed it should, which was previously obtained by another method (orthogonal polynomials). With this, the result passes an important consistency test. The microscopic spectral density becomes (as expected)

$$\rho_S(\tilde{x}) = \lim_{\tilde{y}\to\tilde{x}} K_S(\tilde{x}, \tilde{y}) = \tilde{x}(J_0^2(2\tilde{x}) + J_1^2(2\tilde{x})) \qquad (35)$$

5.2 $T \neq 0$

In the case $T \neq 0$, the integrations can also be separated using a Feynman-parameterization. However, they cannot be evaluated directly, but one has to calculate them in saddlepoint approximation. In the microscopic limit this can be done, at least for the imaginary part of $\hat{K}_N$, in which we are interested. The result is

$$\lim_{N\to\infty}\frac{1}{2N}\hat{K}(\frac{\tilde{x}}{N}, \frac{\tilde{y}}{N}) = i\sqrt{1-t^2}K_S(\sqrt{1-t^2}\tilde{x}, \sqrt{1-t^2}\tilde{y}) \qquad (36)$$

This result is most remarkable: it is of the exact same form as the one at zero temperature, the only difference is a scaling factor in the argument. This is in this case what is meant by "universality" of the result. The introduction of a thermal deformation of the matrix model, which even breaks the symmetry, does not alter the form of the result.

6 Unquenched calculation

The problem in the unquenched case is the fact that the fermion-determinant $\det^{N_f}(D)$ appears unpaired in the generating functional (that is, not divided by another, bosonic, determinant). This means that the supermatrices which have to be introduced to write the determinants as integrations over supervectors have fermionic- and bosonic blocks of different size. In the calculation of the group integrals over the decoupling matrices σ this leads to a Berezinian that can no longer be written as a determinant. Due to this, the correlation functions can no longer be written in terms of the determinant of kernels, and the results of this calculation (which is in principle doable) are in a form which can not easily be handled, much less compared to previous results. But this difficulty can be circumvented.

The unquenched case can be calculated by considering a supplemented generating functional which contains additional determinants with dummy variables, such that the resulting supermatrices once again have fermionic and bosonic blocks of same size. After taking a limit on the dummy variables in such a way that the contribution due to them drops out again, one can obtain the correlation functions in terms of determinants over kernels, which reduce in the appropriate limits (quenching, fermion masses $m_f \to \infty$, zero temperature, $T \to 0$) to the known and previously calculated functions. This calculation was done by B. Seif, T. Wettig and T. Guhr [7].

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