Topics in Random Matrix Theory

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Abstract

1 Spectra of complex systems

1.1 Introduction

Typically when we talk about spectra we think of quantum mechanics. Among the best known examples are the spectrum of a harmonic oscillator with eigenvalues

$$E_n = (n + \frac{1}{2})\hbar\omega \tag{1}$$

and the spectrum of an angular momentum operator with eigenvalues of J^2 given by

$$j(j+1). \tag{2}$$

However, it is the exception that we are able to write down analytical formula for the complete set of eigenvalues of an operator or even a single eigenvalue. For basically all systems that occur in nature this is not possible. One of the questions we wish to address is whether we are able to find any generic features in spectra even if we do not have any detailed analytical knowledge of the eigenvalues.

Let me first give of overview of different systems in which spectra play an important role. One of the most important examples is the Schrödinger equation

$$(-\nabla^2 + V(x))\psi = \lambda\psi, \tag{3}$$

of course with boundary conditions on the wave functions. One particularly simple class of potentials are the billiards where $V(x)$ is zero inside a connected domain and is infinite on its boundary. In particular, billiards in the two-dimensional plane have been studied extensively in the literature. Although it is trivial to solve the Schrödinger equation, the complication arises because of the boundary conditions which can only be satisfied for a discrete set of eigenvalues.

A second example is the Helmholtz equation for the electric and magnetic field inside a resonance cavity

$$(\nabla^2 + k^2)\vec{E} = 0, \qquad (\nabla^2 + k^2)\vec{B} = 0. \tag{4}$$

In this case one can distinguish different types of boundary conditions, transverse electromagnetic, transverse magnetic and transverse electric. One case of special interest are transverse magnetic modes in a two-dimensional cavity. For the E_z we find the wave equation

$$(\nabla^2 + k^2)\vec{E}_z = 0, \tag{5}$$

with the boundary condition that E_z vanishes on the boundary. This is exactly the Schrödinger equation for a two-dimensional billiard.

A third example is that of sound waves in materials. In that case the wave equation is given by

$$(\nabla^2 + k^2)\vec{u} = 0, \tag{6}$$

where $\vec{u}$ can be decomposed in the longitudinal and the transverse displacement. This equation is more complicated than either the Maxwell or Schhrödinger equation because the two types of modes are coupled through reflections at the boundary.

A fourth example is the Dirac operator in QCD. In this case the wave equation is given by

$$\gamma_\mu(\partial_\mu + iA_\mu)\psi = \lambda\psi. \tag{7}$$

Here, the γ_μ are the Euclidean gamma matrices and the A_μ are $SU(N_c)$ valued gauge fields. We will discuss this example in much more detail later in this lecture.

As a fifth example I mention the zero of the Riemann ζ function. This function is defined as

$$\zeta(s) = \sum_n \frac{1}{n^s}. \tag{8}$$

Empirically, it is know that all zero's are on the line $s = \frac{1}{2}$. The question which we wish to discuss later in this lecture is what these zeros have in common with the eigenvalues of complex dynamical system.

All differential operators we have discussed up to now are Hermitian operators. They can be diagonalized by a unitary transformation and have real eigenvalues, i.e.

$$H = U\Lambda U^{-1}. \tag{9}$$

Later we will see applications of non-hermitian differential operators. They can be diagonalized by a similarity transformation with eigenvalues scattered in the complex plane.

In many cases spectra can be observed experimentally. The best-known examples are atomic and nuclear spectra. However, a great deal of spectroscopy has been performed on molecular spectra as well. As a much more recent example I mention quantum dots in which electrons are enclosed inside a heterostructure and collide elastically against the boundary. Another example are the spectra of small metallic particles, where the electron spectra are determined by the boundary conditions on the surface. In this case the specific heat is determined by the spectral properties of the particles.

1.2 Statistical analysis of spectra

We have seen many different examples of spectra. Basically none of them can be obtained analytically. In general, one can obtain many different eigenvalues. This cries out for a reduction of information, and it is therefore natural the perform a statistical analysis of the spectra. The spectral density can be written as

$$\rho(\lambda) = \sum_k \delta(\lambda - \lambda_k) \tag{10}$$

with normalization given by

$$\int d\lambda \rho(\lambda) = N. \tag{11}$$

One frequently employs the integrated spectral density

$$N(E) = \int_{-\infty}^{E} \rho(\lambda) d\lambda. \tag{12}$$

Since $N(E)$ jumps by one at the position of each eigenvalue, it is also know as the staircase function, Generally, the spectral density can be smoothened over a scale that is much larger than the average level spacing. Let us denote the smoothened spectral density by $\bar{\rho}$. Typically, $\bar{\rho}$ depends on the specific properties of the system. Therefore one would like to eliminate this dependence from the spectrum. This is achieved by the so-called unfolding procedure. The unfolded spectrum is given by

$$\lambda_k^{\rm unf} = \int_0^{\lambda_k} \bar{\rho}(\lambda) d\lambda. \tag{13}$$

One can easily verify that the average spacing of the unfolded sequence is equal to unity.

The statistical analysis of the unfolded eigenvalues is performed by a variety of statistics. The simplest statistic is the so called nearest neighbor spacing distribution. It is denoted by $P(S)$ and is just a histogram of neighboring unfolded levels. A second class of statistics are obtained from counting the number of levels in a sequence that contains n levels on average when moving along the spectrum. If the number of levels in consequetive intervals is given by n_k, we can define the moments

$$M_p(n) = \frac{1}{N}\sum_{k=1}^N n_k^p. \tag{14}$$

Of course, we have that $M_1(n) \to n$ for a large sample. The number variance is defined by

$$\Sigma^2(n) = M_2(n) - n^2. \tag{15}$$

Similarly, one can define higher order cumulants.

Another frequently used statistic is the Δ_3 statistic which is defined as

$$\Delta_3(L) = \frac{2}{L^4}\int_0^L (L^3 - 2L^2 r + r^3)\Sigma^2(r) dr. \tag{16}$$

It has the property that quadratic functions are projected to zero by the kernel of this integral operator. Generally, this statistic is much smoother that the number variance and for that reason it has been used widely in the literature.
Exercise. Calculate the number variance and the Δ_3 statistic for a picket fence spectrum, i.e. a spectrum of equally spaced eigenvalues.

1.3 Statistics for uncorrelated eigenvalues

Let us calculate $P(S)$ and $\Sigma_2(n)$ for independently distributed eigenvalues with average level spacing equal to unity. Then $P(S)dS$ is the probability that there are no eigenvalues inside the interval $[0, S]$ and one eigenvalue in $[S, S + dS]$. Dividing the first interval in n equal pieces we find

$$P(S)dS = (1 - \frac{S}{n})^n DS \to e^{-S} dS. \tag{17}$$

The number variance is given by

$$\begin{aligned} \Sigma^2(n) &= \langle \int_0^n \rho(\lambda) d\lambda \int_0^n \rho(\lambda') d\lambda' \rangle - n^2, \\ &= \int_0^n \int_0^n d\lambda d\lambda' \langle \sum_{k,l} \delta(\lambda - \lambda_k)\delta(\lambda' - \lambda_l) \rangle - n^2. \end{aligned} \tag{18}$$

The average factorizes for different eigenvalues. We finally find

$$\Sigma_2(n) = n - \frac{n^2}{N}, \tag{19}$$

where N is the total number of eigenvalues.

The Δ_3-statistic is obtained by a simple integration resulting in

$$\Delta_3(L) = \frac{L}{15}. \tag{20}$$

However, if one calculates $P(S)$ and $\Sigma_2(n)$ for physical spectra, one finds different results for these observables. In particular, one notices a suppression of small spacings and a strongly reduced number variance for large n. These qualities are characteristic for spectra of many complex systems and are known as level repulsion and spectral rigidity, respectively.

1.4 Correlation functions

The number variance can be expressed in terms of the two-point correlation function of the unfolded eigenvalues. The two point correlation function of the not necessarily unfolded eigenvalues is defined by

$$\rho_2(\lambda, \lambda') = \langle \rho(\lambda)\rho(\lambda') \rangle - \langle \rho(\lambda) \rangle \langle \rho(\lambda') \rangle. \tag{21}$$

The nearest neighbor spacing distribution cannot be expressed in a two-point correlator. The reason is that it measures the probability that none of the other eigenvalues are inside the interval $[0, S]$.

The correlation function $\rho_2(\lambda, \lambda')$ includes a term in which the eigenvalues are equal, and can thus be decomposed as

$$\rho_2(\lambda, \lambda') = \delta(\lambda - \lambda')\langle \rho(\lambda) \rangle + R_2(\lambda, \lambda'). \tag{22}$$

The two-point correlation function satisfies the sum rule

$$\int d\lambda \rho_2(\lambda, \lambda') = 0, \tag{23}$$

where the integral is over the complete spectrum.

In the literature one also frequently uses the quantity $Y_2(\lambda, \lambda')$ for the two-point correlation function of the unfolded eigenvalues

$$Y_2(\lambda, \lambda') = -R_2(\lambda, \lambda'), \tag{24}$$

where the minus sign is conventional. If a smoothened average spectral density can be defined, it is natural to expect that Y_2 is translational invariant, i.e.,

$$Y_2(\lambda, \lambda') = Y_2(\lambda - \lambda'). \tag{25}$$

In that case, the number variance can be expressed as

$$\Sigma^2(L) = L - \int_0^L (L - r) Y_2(r) dr. \tag{26}$$

References

[1] T. Guhr, A. Müller-Groeling and H.A. Weidenmüller, Phys. Rept. **299** (1998) 189.

[2] M. Mehta, *Random Matrices*, Academic Press, San Diego 1991.

[3] K.B. Efetov, *Supersymmetry in disorder and chaos*, Cambridge University Press, (1997).

[4] O. Bohigas, M. Giannoni, Lecture notes in Physics **209** (1984) 1.

[5] M. Stephanov, T. Wettig and J.J.M. Verbaarschot, in Encyclopedia of Electronics and Electrical Engineering.

[6] J.J.M. Verbaarschot, Kyoto Lectures.