10 The Supersymmetric Method of Random Matrix Theory: The Two-Point Function

In this chapter we discuss the two-point correlation function for the GUE using the supersymmetric method. The effective partition function is constructed by mainly relying on the symmetries of the generating function. For the calculation of the two-point function we refer to an article by Martin Zirnbauer and myself [6] and a very detailed article by Joe Zuk [7] that is based on a papers by Efetov [1] and Zirnbauer [3].

10.1 Generating Function

The generating function for the two-point function is defined by

$$Z(J_1, J_2) = \int dHP(H) \frac{\det(x + J_1 + i\epsilon - H)}{\det(x - J_1 + i\epsilon - H)} \frac{\det(y + J_2 - i\epsilon - H)}{\det(y - J_2 - i\epsilon - H)}. \tag{1}$$

Here, ϵ is an infinitesimal increment, and $P(H)$ is the Gaussian probability distribution for the unitary ensemble. The correlation function of two resolvents is given by

$$\langle G(x + i\epsilon)G(y - i\epsilon)\rangle = \frac{1}{4N^2} \partial_{J_1}|_{J_1=0} \partial_{J_2}|_{J_2=0} Z(J_1, J_2). \tag{2}$$

The generating function can be written in terms of Gaussian integrals

$$Z(J_1, J_2) = \int d\psi d\psi^* e^{i\phi_1^*(x+J_1+i\epsilon-H)\phi_1 + i\chi_1^*(x-J_1+i\epsilon-H)\chi_1 - i\phi_2^*(y+J_2-i\epsilon-H)\phi_2 - i\chi_2^*(y-J_2-i\epsilon-H)\chi_2}, \tag{3}$$

where the vector ψ is defined by

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \chi_1 \\ \chi_2 \end{pmatrix}. \tag{4}$$

Two widely used matrices are the matrix L, which breaks the symmetry between the 1-space and the 2-space, and the matrix k which breaks the supersymmetry. They are defined by

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5}$$

An important role is played by the energy difference

$$\omega = x - y. \tag{6}$$

For $\omega = 0$ and $J_1 = J_2 = 0$ the partition function (1) is invariant under the transformations

$$\psi \to U\psi, \qquad \psi^* \to \psi^* LU^{-1}L. \tag{7}$$

where $U \in Gl(2|2)$. This symmetry is broken explicitly by the symmetry breaking term

$$i\frac{\omega}{2}(\phi_1^*\phi_1 + \phi_2^*\phi_2 + \chi_1^*\chi_1 + \chi_2^*\chi_2), \tag{8}$$

and spontaneously by the expectation value of

$$\langle \psi_i^* \psi_j \rangle = N\Sigma k_{ij}. \tag{9}$$

This result follows from analysis similar to the discussion of the one-point function. The appearance of the matrix k follows from the differentiation of the generating functions (1) and (3) with respect to x and y. For example, differentiation with respect to a fermionic source term and a bosonic source term differs by a minus sign, i.e.

$$\partial_J \frac{1}{\det(x+J-H} = \partial_J \det(x+J-H). \tag{10}$$

One also would expect a matrix L in the r.h.s. of eq. (9). However, because of the $i\epsilon$ there is an additional relative minus sign between the expectation value of $\phi_1^*\phi_1$ and $\phi_2^*\phi_2$. The symmetry of the partition function is thus broken spontaneously to $Gl(1|1) \times Gl(1|1)$. If the symmetry transformations applied to the symmetry breaking term we obtain

$$\begin{aligned} \frac{i\omega}{2}(LU^{-1}LU)_{ij}\psi_i^*\psi_j \rightarrow & \quad \frac{i\omega}{2}(U^{-1}LUL)_{ij}N\Sigma k_{ij}, \\ = & \quad i\frac{N\omega}{2}\Sigma\mathrm{Trg}(LU^{-1}LU). \end{aligned} \tag{11}$$

We are interested in the thermodynamic limit in which ωN remains fixed. To leading order in $1/N$ the integrals in the partition function can be approximated by the integral over the saddle-point manifold, i.e.

$$Z(J_1, J_2) = \int dQ e^{i\frac{\omega N}{2}\Sigma\mathrm{Trg}(LQ)+J_1 iN\Sigma\mathrm{Trg}kQ_{11}-J_2 iN\Sigma\mathrm{Trg}kQ_{22}}, \tag{12}$$

with the Goldstone manifold is given by

$$Q = U^{-1}LU. \tag{13}$$

The subscripts 11 and 22 refer to projection to the respective blocks of the Q matrices. By differentiation with respect to the source terms we thus obtain

$$\langle G(x+i\epsilon)G(y-i\epsilon)\rangle = \frac{\Sigma^2}{4}\int dQ\mathrm{Trg}(kQ_{11})\mathrm{Trg}(kQ_{22})e^{-\frac{\omega N}{2}\Sigma\mathrm{Trg}(LQ)}. \tag{14}$$

The integral is over the maximum Riemannian submanifold of $Gl(2|2)/Gl(1|1) \times Gl(1|1)$ and the measure is the invariant measure of the unbroken group.

If x and y would be on the same side of the real axis, the matrix L would not appear in the partition function, and the symmetry between the 1-variable and the 2-variables would not be spontaneously broken. In that case there is no Goldstone manifold and, to leading order in $1/N$ the integrals over the 1-variables and 2-variables factorize. Therefore, they only contribute to the disconnected piece of the correlation function.

10.2 The Integration Manifold

In order to identify the integration manifold, we will be guided by the principle of convergence. Grassmann integrals are always convergent, and we can ignore them in the discussion of this point. Then fermion-fermion blocks and the boson-boson blocks decouple and can be treated separately. Let us first consider the quadratic form that enters in the bosonic integrals

$$\begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} L \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{15}$$

Convergence is assured if the symmetry transformation leaves the complex conjugation structure of this quadratic form invariant. For

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to U \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{16}$$

we require that

$$LU^{-1}L = U^\dagger \quad \text{or} \quad U^\dagger L U = L. \tag{17}$$

If we parameterize U as

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{18}$$

we obtain the equations

$$\begin{aligned} a^*a - b^*b &= 1, & (19)\\ c^*c - d^*d &= -1, & (20)\\ ac^* &= bd^*, & (21)\\ ca^* &= db^*. & (22)\\ & & (23) \end{aligned}$$

Combining these equations we find that $|a| = |d|$ and $|b| = |c|$. In our partition function, the matrix U only enters in the combination of the coset ULU^{-1}.Without loss of generality we can omit the phase factors of a and d. Then we obtain $b = c^*$. Our result for the parameterization of U is thus given by

$$U = \begin{pmatrix} \sqrt{1+a^*a} & a \\ a^* & \sqrt{1+a^*a} \end{pmatrix}. \tag{24}$$

Next we consider the fermion-fermion block which enters in the effective partition function as

$$e^{\frac{N\omega}{2}\mathrm{Tr} L Q^{FF}} \tag{25}$$

With U parametrized according to (18) we obtain

$$e^{N\omega(ad+bc)/(ad-bc)}. \tag{26}$$

For non-compact matrices U the exponent cannot be made negative definite. Convergence of the integrals then forces us to parameterize U according to the compact subgroup of $Gl(2)$ which is

$U(2)$. Because of the coset structure we have that $U \in U(2)/U(1) \times U(1)$. A parameterization of an element is this coset is given by

$$\begin{pmatrix} \cos\theta & ie^{i\phi}\sin\theta \\ ie^{-i\phi}\sin\theta & \cos\theta \end{pmatrix}. \tag{27}$$

More suggestively this can be written as

$$U = \begin{pmatrix} \sqrt{1-b^*b} & ib \\ ib^* & \sqrt{1-b^*b} \end{pmatrix}, \tag{28}$$

with the understanding that the branches of the square root are to be chosen according to a parameterization with a cosine. We observe that the parameterization of the fermion-fermion variable is structurally the same as the parameterization of the boson-boson variables.

Next we consider the Grassmann variables. A matrix $U \in Gl(2|2)$ contains 8 Grassmannian entries. However, in the coset U four of them can be absorbed by the $Gl(1|1) \times Gl(1|1)$ subgroup. Our final result for the parameterization of U is then given by

$$U = \begin{pmatrix} \sqrt{1+t_{12}t_{21}} & t_{12} \\ t_{21} & \sqrt{1+t_{21}t_{12}} \end{pmatrix}. \tag{29}$$

with

$$t_{12} = \begin{pmatrix} a & \alpha \\ \beta & ib \end{pmatrix}, \qquad t_{21} = \begin{pmatrix} a^* & \gamma \\ \delta & ib^* \end{pmatrix}. \tag{30}$$

10.3 The Invariant Measure

Now that we have found a parameterization of the integration manifold we are able to calculate the invariant measure. From measure theory we know that the invariant measure is given by

$$\operatorname{detg}\left(\frac{\delta(\{U^{-1}dU\}_{12,21})}{\delta t_{12}\delta t_{21}}\right) dt_{12}dt_{21}. \tag{31}$$

Instead of calculating the graded determinant for the general we hope to convince the reader that it is equal to unity by evaluating it for two special cases.
Example 10.1. We consider the bosonic matrix

$$U = \begin{pmatrix} \sqrt{1+a^*a} & a \\ a^* & \sqrt{1+a^*a} \end{pmatrix}. \tag{32}$$

In order to calculate the invariant measure on the space of U matrices we have to evaluate

$$U^{-1}dU = \begin{pmatrix} \frac{1}{2}(a^*da - ada^*) & \sqrt{1+a^*a}da - \frac{a}{2\sqrt{1+a^*a}}(ada^* + a^*da) \\ \sqrt{1+a^*a}da^* - \frac{a^*}{2\sqrt{1+a^*a}}(ada^* + a^*da) & \frac{1}{2}(ada^* - a^*da) \end{pmatrix}. \tag{33}$$

The Jacobian for the transformation from the variables $(U^{-1}dU)_{12}$ and $(U^{-1}dU)_{21}$ to the variables a and a^* is given by the determinant of the matrix

$$\begin{pmatrix} \sqrt{1+a^*a} - \frac{a^*a}{2\sqrt{1+a^*a}} & -\frac{a^2}{2\sqrt{1+a^*a}} \\ -\frac{a^{*2}}{2\sqrt{1+a^*a}} & \sqrt{1+a^*a} - \frac{a^*a}{2\sqrt{1+a^*a}} \end{pmatrix}, \tag{34}$$

which is easily seen to be equal to unity. The invariant measure is thus simply given by $dada^*$.
Example 10.2. Next we consider a more complicated example with U_F given by

$$U_F = \begin{pmatrix} 1+\frac{\alpha\delta}{2} & 0 & 0 & \alpha \\ 0 & 1+\frac{\beta\gamma}{2} & \beta & 0 \\ 0 & \gamma & 1+\frac{\gamma\beta}{2} & 0 \\ \delta & 0 & 0 & 1+\frac{\delta\alpha}{2} \end{pmatrix}. \tag{35}$$

For $U_F^{-1} dU_F$ we obtain

$$\begin{pmatrix} -\frac{1}{2}(\alpha d\delta + \delta d\alpha) & 0 & 0 & d\alpha \\ 0 & -\frac{1}{2}(\gamma d\beta + \beta d\gamma) & d\beta & 0 \\ 0 & d\gamma & -\frac{1}{2}(\gamma d\beta + \beta d\gamma) & 0 \\ d\delta & 0 & 0 & -\frac{1}{2}(\alpha d\delta + \delta d\alpha) \end{pmatrix}. \tag{36}$$

One observes that the Jacobian for the transformation from the 12 and 21 blocks of $U_F^{-1} dU_F$ to α, β, γ and δ is equal to unity. The invariant measure is therefore given by $d\alpha d\beta d\gamma d\delta$.
Exercise 10.1. Calculate the Berezinian for the general case and show that its value is equal to unity as well . The invariant measure is thus given by $dt_{12} dt_{21}$.

The measure in terms of the t_{12} variables is flat with invariant line element given by

$$ds^2 = \mathrm{Trg} dt_{12} dt_{21}. \tag{37}$$

The measure is therefore invariant under the transformations

$$t_{12} \to u t_{12} v^{-1}, \qquad t_{21} \to v t_{21} u^{-1}. \tag{38}$$

10.4 Transformation to Polar Variables.

Although an explicit calculation of the Grassmann integrations using the parameterization of the previous section is possible, the algebra involved in such calculation is quite laborious. For that reason we introduce new integration variables defined by the polar decomposition

$$t_{12} = u \begin{pmatrix} \mu_1 & 0 \\ 0 & i\mu_2 \end{pmatrix} v^{-1}, \tag{39}$$

and

$$t_{21} = v \begin{pmatrix} \mu_1^* & 0 \\ 0 & i\mu_2^* \end{pmatrix} u^{-1}, \tag{40}$$

with u and v given by

$$u = \begin{pmatrix} 1+\frac{\alpha\beta}{2} & \alpha \\ \beta & 1-\frac{\alpha\beta}{2} \end{pmatrix}, \qquad v \begin{pmatrix} 1+\frac{\gamma\delta}{2} & \gamma \\ \delta & 1-\frac{\gamma\delta}{2} \end{pmatrix}. \tag{41}$$

The matrices u and v satisfy the property that

$$u\sigma_3 u = \sigma_3, \qquad v\sigma_3 v = \sigma_3, \tag{42}$$

with σ_3 the third Pauli-matrix.

To calculate the Berezinian of the polar variables we consider the variations

$$u^{-1}\delta t_{12} v = u^{-1}\delta u \mu + \delta\mu - \mu v^{-1}\delta v, \tag{43}$$
$$v^{-1}\delta t_{21} u = v^{-1}\delta v \mu^* + \delta\mu^* - \mu^* u^{-1}\delta u, \tag{44}$$
$$\tag{45}$$

where μ and μ^* are diagonal matrices with diagonal elements μ_1 and $i\mu_2$, and μ_1^* and $i\mu_2^*$, respectively. The block of the Berezinian corresponding to differentiation of the boson-fermion blocks of $u^{-1}\delta t_{12}$ and $v^{-1}\delta t_{21} u$ with respect to μ and μ^* is equal to zero. The determinant of the block corresponding to differentiation of the boson-boson block with respect to these variables is equal to one. The only nontrivial contribution to the Berezinian then arises from the variation of the boson-fermion blocks of $u^{-1}\delta t_{12}$ and $v^{-1}\delta t_{21} u$ with respect to the off-diagonal elements of $u^{-1}du$ and $v^{-1}dv$. The Berezinian from the t_{12} and t_{21} variables to the polar variables is thus given by the inverse determinant of

$$\begin{pmatrix} i\mu_2 & 0 & -\mu_1 & 0 \\ 0 & \mu_1 & 0 & -i\mu_2 \\ -\mu_1^* & 0 & i\mu_2^* & 0 \\ 0 & -i\mu_2^* & 0 & \mu_1^* \end{pmatrix}. \tag{46}$$

The value of the Berezinian is thus equal to

$$B = \frac{1}{(\mu_1^*\mu_1 + \mu_2^*\mu_2)^2}. \tag{47}$$

To derive the final result for the two-point function we also need

$$\mathrm{Trg} k Q_{11} = \lambda_1 + \lambda_2 + 2\alpha\beta(\lambda_1 - \lambda_2), \tag{48}$$
$$\mathrm{Trg} k Q_{22} = -\lambda_1 - \lambda_2 - 2\gamma\delta(\lambda_1 - \lambda_2), \tag{49}$$

where λ_1 and λ_2 are the diagonal elements of Q_{11} given by

$$\lambda_1 = 1 + 2\mu_1^2, \tag{50}$$
$$\lambda_2 = 1 - 2\mu_2^2. \tag{51}$$

For the exponent in the partition function we obtain

$$e^{iN\omega\Sigma(\lambda_1-\lambda_2)}. \tag{52}$$

After doing the angular integrations, the final result for the two-point correlation function is given by

$$\begin{aligned} \langle G(x+i\epsilon)G(y-i\epsilon)\rangle &= \frac{\Sigma^2}{16}\int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda_2 \frac{d\alpha d\beta d\gamma d\delta}{(\lambda_1-\lambda_2)^2} \\ &\times (\lambda_1+\lambda_2+2\alpha\beta(\lambda_1-\lambda_2))(\lambda_1+\lambda_2+2\gamma\delta(\lambda_1-\lambda_2))e^{-N\omega\Sigma(\lambda_1-\lambda_2)} \end{aligned} \tag{53}$$

According toe the Efetov-Zirnbauer theorem, the terms of second order in the Grassmann variables do not contribute to the final result. Because of the extra factor $\lambda_1 - \lambda_2$ there are no problems with singularities. However, the term of zeroth order in the Grassmann variables is the product of a divergent result, from the integration over the bosonic variables, and zero from the Grassmann integrations. In this case there will be a finite contributions due to the Efetov-Wegner terms, which will emerge from a careful treatment of the singularity.

10.5 Final Result and Efetov-Wegner Terms

We regularized our integrals by excluding a small region around the singular region $a = b = 0$. Using the procedure of re. [4] we replace the integration measure by

$$dada^* dbdb^* \to dada^* dbdb^* \, \theta(\sqrt{a^*a + b^*b} - \rho), \tag{54}$$

and take the limit $\rho \to 0$ at the end of the calculation. We express this regularized measure in terms of the polar variables μ_1, μ_1^*, μ_2 and μ_1^*. Some algebra using the explicit polar decomposition of t_{12} and t_{21} results in

$$\begin{aligned} a^*a + b^*b = & \quad \mu_1^*\mu_1 + \mu_2^*\mu_2 + (\mu_1^*\mu_1 - \mu_2^*\mu_2)(\alpha\beta + \gamma\delta) + 2i\mu_1\mu_2^*\beta\gamma + 2i\mu_1^*\mu_2\delta\alpha \\ & + \quad \frac{3}{2}\alpha\beta\gamma\delta(\mu_1^*\mu_1 + \mu_2^*\mu_2), \end{aligned} \tag{55}$$

For $\sqrt{a^*a + b^*b}$ we obtain

$$\sqrt{a^*a + b^*b} = R + \frac{1}{2R}\left[(\mu_1^*\mu_1 - \mu_2^*\mu_2)(\alpha\beta + \gamma\delta) + 2i\mu_1\mu_2^*\beta\gamma + 2i\mu_1^*\mu_2\delta\alpha\right] + \frac{R}{2}\alpha\beta\gamma\delta, \tag{56}$$

where

$$R = \sqrt{\mu_1^*\mu_1 + \mu_2^*\mu_2} = \sqrt{\lambda_1 - \lambda_2}. \tag{57}$$

Keeping only the terms of zeroth and maximum order in the Grassmann variables the θ function can be expanded as

$$\theta(\sqrt{a^*a + b^*b} - \rho) = \theta(R - \rho) + \frac{R}{2}\delta(R - \rho)\alpha\beta\gamma\delta + \frac{R^2}{4}\delta'(R - \rho)\alpha\beta\gamma\delta. \tag{58}$$

The derivative of the delta function can be converted into a delta function by partial integration,

$$\frac{1}{R^2}\partial_R 2R\delta(R^2 - \rho^2) \to \frac{4}{R^2}\delta(R^2 - \rho^2), \tag{59}$$

and give the same contribution as the delta function.

After performing the Grassmann integration and including the Efetov-Wegner terms, the correlation function (53) becomes

$$\langle G(x + i\epsilon)G(y - i\epsilon)\rangle = \frac{\Sigma^2}{16}\int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda_2 \left[\frac{2\delta(\lambda_1 - \lambda_2 - \rho^2)}{\lambda_1 - \lambda_2}(\lambda_1 + \lambda_2)^2 + 4\right] e^{-N\omega\Sigma(\lambda_1 - \lambda_2)}. \tag{60}$$

The first integral can be performed as follows

$$\int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda_2 \frac{2\delta(\lambda_1 - \lambda_2 - \rho^2)}{\lambda_1 - \lambda_2)}(\lambda_1 + \lambda_2)^2 = \int_{1-\rho^2}^1 d\lambda_2 \frac{2(\rho^2 + 2\lambda_2)^2}{\rho^2} = 16, \tag{61}$$

where we have taken the limit $\rho \to 0$. The final result for the correlation function is given by

$$\langle G(x + i\epsilon)G(y - i\epsilon)\rangle = \Sigma^2 + \frac{\Sigma^2}{4}\int_1^\infty d\lambda_2 \int_{-1}^1 d\lambda_2 e^{iN\omega\Sigma(\lambda_1 - \lambda_2)} = \Sigma^2 - i\frac{1}{2}\frac{\sin(N\omega\Sigma)e^{iN\omega\Sigma}}{(\omega N)^2}. \tag{62}$$

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