11 Itzykson-Zuber integrals and the Duistermaat-Heckman theorem

In this chapter we discuss Harish-Chandra-Itzykson-Zuber integrals which, in the case of the $SU(N)$ groups are given by [1]

$$\int dU e^{i\mathrm{Tr} XU^{-1}YU} = c\frac{\det e^{ix_k y_l}}{\Delta(x)\Delta(y)}. \tag{1}$$

Here, the integral is over $SU(N)$ and X and Y are Hermitean matrices with eigenvalues x_i and y_i and c is a normalization constant. The Vandermonde determinants are given by

$$\Delta(x) = \prod_{k<l}(x_k - x_l). \tag{2}$$

The general case of this remarkable formula was first derived by Harish-Chandra [5]. Below we will derive this result in two different ways. First, via a diffusion equation, and, second, by means of a saddle-point approximation. Remarkably, the saddle-point result turns out to be exact. This is an example of the Duistermaat-Heckman [4] theorem which states under which conditions a saddle-point approximation becomes exact.

11.1 Derivations of the Itzykson-Zuber integral.

We consider the integral

$$\xi(A,t) = \frac{1}{(2\pi t)^{-n^2/2}} intdBe^{-\frac{1}{2t}\mathrm{Tr}(A-B)^2}\eta(B), \tag{3}$$

with A and B Hermitean matrices, and $\eta(B)$ is an invariant function of B. By expression the trace in components one can easily show that ξ satisfies the diffusion equation

$$\partial_t \xi = \frac{1}{2}\nabla_A^2 \xi, \tag{4}$$

with initial condition

$$\xi(A,0) = \eta(A). \tag{5}$$

The matrices A and B can be diagonalized by a unitary transformation

$$A = UXU^{-1}, \qquad B = VYV^{-1}. \tag{6}$$

The nonzero matrix elements of the diagonal matrices X and Y will be denoted by x_i and y_i, respectively. The Jacobian from the A variables to the U and X variables is given by

$$dA = \Delta^2(X) dX d\Omega_A. \tag{7}$$

where the angular degrees of freedom arec contained in Ω_A. In terms of these coordinates the Laplacian is given by

$$\nabla_A^2 = \frac{1}{\Delta^2(x)}\partial_{x_i}\Delta^2(x)\partial_{x_i} + \nabla_{\Omega_A}^2. \tag{8}$$

For an invariant function $\eta(B)$, the function $\xi(A,t)$ depends only on the eigenvalues of A and thus satisfies the equation

$$\partial_t \xi = \frac{1}{2}\frac{1}{\Delta^2(x)}\partial_i \Delta^2(x)\partial_i \xi \tag{9}$$

with initial condition

$$\xi(X,0) = \eta(X). \tag{10}$$

To solve this equation we make the substitution

$$F(X,t) = \Delta(x)\xi(X,t). \tag{11}$$

The second derivative of F is given by

$$\partial_i^2 F = \Delta \partial_i^2 \xi + 2\partial_i \Delta \partial_i \xi + \xi \partial_i^2 \Delta. \tag{12}$$

The r.h.s. of this equation can be rewritten as

$$\Delta \frac{1}{\Delta^2}\partial_i(\Delta^2 \partial_i \xi). \tag{13}$$

This results in the diffusion equation

$$\partial_t F = \frac{1}{2}\partial_i^2 F. \tag{14}$$

The solution of this diffusion equation is given by

$$F(X,t) = c\frac{1}{t^{n/2}}\int dY \Delta(y) e^{-\frac{1}{2t}\sum_i (x_i - y_i)^2}\eta(Y), \tag{15}$$

with c an arbitrary constant. This result should be valid for any invariant function of the eigenvalues. An invariant function is also a symmetric function of the eigenvalues. The exponent in this equation can be factorized in symmetric functions of the integration variables and $\exp(\sum_i x_i y_i/t)$. Only the antisymmetrized part of the latter term contributes to the integral. Therefore, we have to antisymmetrize the exponential function before equating the coefficients of the $\eta(B)$. After multiplying out the squares in the exponent of both sides of the equation we obtain

$$\int dU e^{-\frac{1}{2t}\mathrm{Tr} UXU^{-1}VYV^{-1}} = ct^{(n^2-n)/2}\frac{\det e^{-\frac{1}{2t}x_k y_l}}{\Delta(y)\Delta(x)}, \tag{16}$$

which is the final result for the Itzykson-Zuber integral.

11.2 Semiclassical calculation of the Itzykson-Zuber integral.

In this section we calculate the Itzykson-Zuber integral by means of a saddle-point approximation. The saddle-point equations are given by

$$d\mathrm{Tr} XU^{-1}YU = 0. \tag{17}$$

Using the product rule and $dU^{-1} = -U^{-1}dUU^{-1}$ we obtain

$$\mathrm{Tr}XU^{-1}dUU^{-1}YU = \mathrm{Tr}XU^{-1}YUdU. \tag{18}$$

By equating the coefficient of dU on both sides of the equation we obtain

$$XU^{-1}YU = U^{-1}YUX. \tag{19}$$

The solution of this equation is a permutation matrix, i.e.

$$UXU^{-1} = X_\pi, \tag{20}$$

where X_π is the diagonal matrix with diagonal elements $x_{\pi(i)}$. Of course we have

$$U^{-1}YU = Y_{\pi^{-1}}, \tag{21}$$

which immediately proves our assertion that permutation matrices are solutions of our saddle-point equations. From the definition of the integral it is clear that $U \in U(N)/U^N(1)$, and saddle-points that only differ by a diagonal unitary matrix have not to be taken into account.

Let us next expand around the saddle-point,

$$U = \bar{U}e^{i\phi} = \bar{U}(1 + i\phi - \frac{1}{2}\phi^2 + cdots), \tag{22}$$

where $\phi^\dagger = \phi$. To second order in ϕ we obtain

$$\begin{aligned} \mathrm{Tr}XU^{-1}Y &= \mathrm{Tr}X\bar{U}^{-1}Y\bar{U} \\ &+ -\frac{1}{2}\mathrm{Tr}X\phi^2\bar{U}^{-1}Y\bar{U} - \frac{1}{2}\mathrm{Tr}X\bar{U}^{-1}Y\bar{U}\phi^2 + \mathrm{Tr}X\phi\bar{U}^{-1}Y\bar{U}\phi. \end{aligned} \tag{23}$$

The terms linear in ϕ vanish because of the saddle-point equations. Notice that $U^{-1}Y\bar{U}$ and X are both diagonal and commute. In terms of components the terms of second order in ϕ can be written as

$$-\sum_{i<k} \phi_{ik}\phi_{ik}^*(Y_{\pi(i)} - Y_{\pi(k)})(X_i - X_k), \tag{24}$$

where we have used the Hermiticity of ϕ. Summing over all saddle-points we obtain

$$\sum_\pi \frac{e^{\sum_i X_i Y_{\pi(i)}}}{\prod_{i<k}(X_i - X_k)(Y_{\pi(i)} - Y_{\pi(k)})}. \tag{25}$$

If we use that

$$\prod_{i<k}(Y_{\pi(i)} - Y_{\pi(k)}) = (-1)^\pi \prod_{i<k}(Y_i - Y_k), \tag{26}$$

we obtain our desired result

$$\int dU e^{i\mathrm{Tr}XU^{-1}YU} = c\frac{\det e^{x_k y_l}}{\Delta(X)\ \Delta(Y)}. \tag{27}$$

It was first pointed out by Mike Stone [3], that the exact result coincides with the result obtained by a saddle-point approximation.

11.3 Invariant Measure for $SU(2)$

We first calculate the invariant measure of $SU(2)$ for the parametrization

$$U = \frac{1}{\sqrt{1+m^2+r^2+s^2}} \begin{pmatrix} 1+im & -r-is \\ r-is & 1-im \end{pmatrix}. \tag{28}$$

If we write

$$U^{-1}dU = \begin{pmatrix} idm' & dm_1 \\ -dm_1^* & idm' \end{pmatrix}, \tag{29}$$

the invariant measure is given by

$$dm' dm_1 dm_1^*. \tag{30}$$

Next we calculate $U^{-1}dU$

$$U^{-1}dU = \frac{1}{R^2} \begin{pmatrix} (1-im)idm + (r+is)(dr-ids) & (1-im)(-dr-ids) - idm(r+is) \\ (-r+is)idm + (1+im)(dr-ids) & (-r+is)(-dr-ids) - idm(1+im) \end{pmatrix} \tag{31}$$

where

$$R = \sqrt{1+m^2+r^2+s^2}. \tag{32}$$

One easily derives that

$$dm' = \frac{1}{R^2}(dm + sdr - rds), \tag{33}$$

$$dm_1 = \frac{1}{R^2}((r+is)idm + (1-im)(dr+ids)), \tag{34}$$

$$dm_1^* = \frac{1}{R^2}(-(r-is)idm + (1+im)(dr-ids)). \tag{35}$$

The Jacobian from the m', m_1 and m_1^* to the m, r and s variables is then given by

$$J = \frac{1}{R^6} \det \begin{pmatrix} 1 & s & -r \\ ir-s & 1-im & m+i \\ -ir-s & 1+im & m-i \end{pmatrix} = \frac{-2i}{R^4}. \tag{36}$$

The invariant measure in the parametetrization (**??**) is thus given by

$$\frac{2dmdrds}{(1+m^2+r^2+s^2)^2}. \tag{37}$$

Next let calculate the invariant measure for the parametrization

$$U = \begin{pmatrix} e^{i\alpha}\cos\theta & -\sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & e^{-i\alpha}\cos\theta \end{pmatrix}. \tag{38}$$

We have to calculate the Jacobian from the mrs-variables to the angular variables. One easily derives the following relations between these two sets of variables

$$\sin\theta = \frac{\sqrt{r^2+s^2}}{\sqrt{1+m^2+r^2+s^2}}, \qquad \cos\theta = \frac{\sqrt{1+m^2}}{\sqrt{1+m^2+r^2+s^2}}, \tag{39}$$

$$\cos\alpha = \frac{1}{\sqrt{1+m^2}}, \qquad e^{i\phi} = \frac{r-is}{r+is}, \qquad e^{i\alpha} = \frac{1+im}{\sqrt{1+m^2}}. \tag{40}$$

The old and new differentials are related by

$$
\begin{aligned}
d\theta &= \frac{\sqrt{1+m^2}rdr}{R^2\sqrt{r^2+s^2}} + \frac{\sqrt{1+m^2}sds}{R^2\sqrt{r^2+s^2}}, && (41)\\
d\phi &= \frac{sdr - rds}{r^2+s^2}, && (42)\\
d\alpha &= \frac{dm}{1+m^2}. && (43)
\end{aligned}
$$

The Jacobian is then given by

$$
J = \det\begin{pmatrix} \frac{r\sqrt{1+m^2}}{R^2\sqrt{r^2+s^2}} & \frac{s\sqrt{1+m^2}}{R^2\sqrt{r^2+s^2}} & 0 \\ \frac{s}{r^2+s^2} & -\frac{r}{r^2+s^2} & 0 \\ 0 & 0 & \frac{1}{1+m^2} \end{pmatrix} = \frac{1}{R^2\sqrt{(1+m^2)(r^2+s^2)}}. \tag{44}
$$

The invariant measure is then given by the ratio of the invariant measure (**??**) and this Jacobian

$$
\frac{2}{R^4}\frac{1}{J} = 2\cos\theta\sin\theta. \tag{45}
$$

11.4 The Itzykson-Zuber integral for $SU(2)$

In the parametrization (**??**, the exponent in the Itzykson-Zuber integral is given by

$$
\mathrm{Tr}UXU^{-1}Y = x_1y_2 + x_2y_1 + (x_1-x_2)(y_1-y_2)\cos^2\theta. \tag{46}
$$

This results in the Itzukson-Zuber integral

$$
I = \int_0^\pi d\theta \int_0^{2\pi} d\phi 2|\cos\theta\sin\theta| e^{x_1y_2+x_2y_1+(x_1-x_2)(y_1-y_2)\cos^2\theta}. \tag{47}
$$

The integral over ϕ is trivial. The integrand has critical points at $\theta = 0$, $\theta = \pi/2$ and $\theta = \pi$. The integrand is a total derivative on each of these intervals separately, but cannot be written as a total derivative globally. The result of the integral is thus given by

$$
\frac{2(e^{x_1y_1+x_2y_2} - e^{x_1y_2+x_2y_1})}{(x_1-x_2)(y_1-y_2)}, \tag{48}
$$

which is the correct result for the for the Itzykson-Zuber integral. We observe that the integral is localized at the critical points which coincide with the saddle-points of the integrand.

11.5 Connection with the Duistermaat-Heckman Theorem.

It is no accident that the saddle-point approximation for the Itzykson-Zuber integral gave the correct result. The general conditions for the semi-classical exactness of phase-space integrals were outlined in a theorem by Duistermaat and Heckman. The simplest example of this theorem is given by the integral

$$
-\int_0^\pi d\theta \int_0^{2\pi} d\phi \sin\theta e^{a\cos(\theta)} = \frac{2\pi}{a}(e^a - e^{-a}). \tag{49}
$$

The exact answer for this integral follows immediately by observing that the integrand is a total derivative. However, the result of this integral is entirely localized on the saddle-points and is therefore semiclassical exact. Because the integrand vanishes on the saddle-points, the saddle-point integration has to be performed with care and is given by

$$2\pi \int_0^{\infty} \theta e^{a-a\theta^2/2} + \int_{-\infty}^{\pi} (\pi-\theta) e^{-a+a(\pi-\theta)^2/2} = \frac{2\pi}{a}(e^a - e^{-a}). \tag{50}$$

Since the integrand is localized at the critical points it suffices the expand the integrand up to second order at these points.

Both this example and the Itzykson-Zuber integral for $SU(2)$ are of the general type

$$I = \int d\theta |f'(\theta)| e^{f(\theta)}. \tag{51}$$

Locally the integrand is a total derivative and the contributions are localized on the critical points, i.e. on the points where the Jacobian vanishes. The result of the integral is given by

$$I = 2e^{f(\theta_0)} - 2e^{f(\theta_1)} + 2e^{f(\theta_2)} - \cdots. \tag{52}$$

However the points $f(\theta_i) = 0$ are also saddle-points. Expanding to second order around the saddle-points we obtain

$$I = \sum_i \int d\theta \alpha |\theta - \theta_i| e^{f(\theta_i) - \frac{1}{2}(\theta-\theta_i)^2 f''(\theta_i)}, \tag{53}$$

where $f(\theta) = \alpha(\theta - \theta_i)$ in the neigborhood of a critical point. The saddle-point result for the integral is thus given by

$$2\frac{f'(\theta_0)}{f''(\theta_0)} - 2\frac{f'(\theta_1)}{f''(\theta_1)} + 2\frac{f'(\theta_2)}{f''(\theta_2)} - \cdots. \tag{54}$$

We observe that the saddle-point integration is exact if $\alpha = f''(\theta_i)$. One can easily verify that this condition is satisfied for the examples considered above.

In general we have an integrand that can locally be written as a total derivative. However is we exclude the critical points in the integration domain the integrand can be written as a total derivative globally.

$$I = \int_{C_\epsilon} d\omega = \sum_{C_\epsilon} \int_{\partial C_\epsilon} \omega, \tag{55}$$

where the C_ϵ are spheres of radius ϵ around the critical points. The integrand is therefore localized at the critical points. At these points the saddle-point approximation becomes exact. In coordinates where the mixed derivatives vanish the condition for the exactness of the saddle-point approximation is that the ratio of the product of the α_i and the product of the second derivatives is equal to unity.

As an example let us again consider

$$\int sin\theta d\theta d\phi e^{a\cos\theta} = -\frac{1}{a}\int d(d\phi e^{a\cos\theta}) = -\frac{1}{a}\int_{\partial} C(0,\epsilon) d\phi e^{a\cos\theta} - \frac{1}{a}\int_{\partial} C(\pi,\epsilon) d\phi e^{a\cos\theta} \tag{56}$$

Taking into account that the orientation of C_ϵ is opposite on both sided of the sphere, we again find the same final answer. The reason that an infinitesimal integration interval gives a finite ansewere is that ϕ varies by a finite amount (2π) around this circle. This happens only at the North pole and the South pole of the sphere.

For the case of $SU(2)$ we also have a critical point at $\theta = \pi/2$. The reason is that at this point the unitary martrices become independent of α.

References

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