12 Itzykson-Zuber Integrals for Arbitrary Complex Matrices

In this chapter we explore integrals of the type

$$\int dU dV e^{\mathrm{ReTr} U^\dagger \sigma V \rho}, \tag{1}$$

where σ and ρ are arbitrary complex $N_1 /times N_2$ matrices. For definiteness we choose $N_1 > N_2$. The integral is over $U(N)/timesU(N)/U^N(1)$ with the measure given by the invariant measure. Such integrals were considered in papers by Guhr and Wettig [1] and by Jackson, Sener and Verbaarschot [2].

The polar decomposition of the matrix σ is given by

$$\sigma = U_s^\dagger S V_s, \tag{2}$$

where the matrix-elements S_{ii} are the only non-zero matrix elements of S, and U and V are unitary matrices. The matrix ρ has a similar polar decomposition

$$\rho = U_r^\dagger R V_r, \tag{3}$$

but now R has $N_1 - N_2$ zero columns. Because of the invariance of the measure we can assume without loss of generality that σ and ρ are diagonal matrices of the form R and S.

12.1 Calculation of the Integral

In order to calculate the integral (1) we consider the function

$$F(\rho, t) = \frac{1}{(\pi t)^{(N_1 N_2)}} \int d[\sigma] \exp\left(-\frac{1}{t}\mathrm{tr}[(\sigma-\rho)^\dagger(\sigma-\rho)]\right)\varphi(\sigma) \tag{4}$$

with integration measure $d[\sigma] = \prod_{m=1}^{N_1}\prod_{n=1}^{N_2} d\mathrm{Re}\sigma_{mn} d\mathrm{Im}\sigma_{mn}$, and $\varphi(\sigma)$ an invariant function of σ which only depends on the diagonal elements of S. From the invariance of the measure it then follows that $F(\rho, t)$ is an invariant function of ρ as well. The function $F(\rho, t)$ satisfies the diffusion equation

$$\sum_{m=1}^{N_1}\sum_{n=1}^{N_2} \frac{\partial^2}{\partial\rho_{mn}\partial\rho_{mn}^*} F(\rho, t) = \frac{\partial}{\partial t} F(\rho, t) \ . \tag{5}$$

In order to proceed, we express both the Laplacian and the integration measure in the 'polar coordinates' introduced by the diagonalization (2),

$$d[\sigma] = \Omega d[S]\, j^2(S)\, d\mu(U, V) \tag{6}$$

where $d[S] = \prod_{n=1}^{N_2} ds_n$. Here, the constant Ω, which depends on the convention adopted for the measure of the group, will be fixed later. The Jacobian is given by $J(S) \equiv j^2(S)$ with

$$j(S) = \prod_{n=1}^{N_2} s_n^{(N_1-N_2)+1/2} \Delta(\hat{S}^2) \ , \tag{7}$$

and $\Delta(\hat{S}^2)$ is the Vandermonde determinant

$$\Delta(S^2) = \prod_{k<l}(s_k^2 - s_l^2). \tag{8}$$

Because F is an invariant function, it satisfies a diffusion equation which involves only the radial part of the Laplacian:

$$\sum_{n=1}^{N_2} \frac{1}{J(R)} \frac{\partial}{\partial r_n} J(R) \frac{\partial}{\partial r_n} F = 4\frac{\partial F}{\partial t} \quad . \tag{9}$$

The first derivatives of F in this equation can be eliminated by the the introduction of reduced wave function,

$$f(R,t) = j(R)F(R,t) \quad . \tag{10}$$

Now, (9) reduces to

$$\sum_{n=1}^{N_2} \left(\frac{\partial^2}{\partial r_n^2} - \frac{1}{j(R)} \left[\frac{\partial^2}{\partial r_n^2} j(R) \right] \right) f(R,t) = 4\frac{\partial f}{\partial t} \quad , \tag{11}$$

with initial condition that

$$f(R, t \to 0) = j(R)\varphi(R). \tag{12}$$

The Laplacian of $j(R)$ is given by

$$\sum_n \partial_{r_n}^2 j(R) = (\nu + \frac{1}{2})(\nu - \frac{1}{2}) \sum_n \frac{1}{r_n^2} j(R) + 2(\nu + \frac{1}{2}) \prod_n r_n^{\nu+\frac{1}{2}} \sum_n \frac{1}{r_n} \partial_{r_n} \Delta(r_k^2) + \prod_n r_n^{\nu+\frac{1}{2}} \sum_n \partial_{r_n}^2 \Delta(r_k^2). \tag{13}$$

The first derivative of the Vandermonde determinant can be written as

$$\sum_n \frac{1}{r_n} \partial_{r_n} \Delta(r_k^2) = \Delta(r_k^2) \sum_{n \neq l} \frac{1}{r_n^2 - r_l^2} = 0. \tag{14}$$

Using the same argument we find that the second derivative sum up to zero if they act on the same factor of the Vandermonde determinant. The only remaining term is where the second derivative acts on different factors of the Vandermonde determinant. The result is given by

$$\sum_{n,k,l \text{ different}} \frac{4r_n^2}{(r_n^2 - r_k^2)(r_n^2 - r_l^2)} = 0, \tag{15}$$

which follows by combining all terms involving the same triplet of the r_k's and by using the identity

$$\frac{4r_n^2}{(r_n^2 - r_k^2)(r_n^2 - r_l^2)} + \frac{4r_k^2}{(r_n^2 - r_l^2)(r_k^2 - r_n^2)} + \frac{4r_l^2}{(r_l^2 - r_n^2)(r_l^2 - r_k^2)} = 0. \tag{16}$$

Remarkably, we find a separable differential equation

$$\sum_{n=1}^{N_2} \left(\frac{\partial^2}{\partial r_n^2} - \frac{4\nu^2 - 1}{4} \frac{1}{r_n^2} \right) f(R,t) = 4\frac{\partial f}{\partial t} \quad . \tag{17}$$

Because of the presence of the factor $j(R)$ in (10), $f(R,t)$ is an antisymmetric function of the eigenvalues. Therefore, the solution of (17) with initial condition (12) is given by an integral over a Slater determinant

$$f(R,t) = \int d[S] \, \frac{1}{N_2!} \det_{k,l} |g(r_k, s_l; t)| \, j(S) \, \varphi(S) \ , \tag{18}$$

where the kernel $g(r,s;t)$ satisfies the differential equation

$$\frac{\partial^2}{\partial r^2} g - \frac{4\nu^2 - 1}{4r^2} g - 4\frac{\partial g}{\partial t} = 4\delta(r-s)\delta(t) \ . \tag{19}$$

By integrating this equation over t one finds that

$$lim_{t\to 0} g(r,s;t) = \delta(r-s), \tag{20}$$

so that $f(R,t)$ satisfies the correct boundary condition. The kernel can be expressed in terms of the regular eigenfunctions of the Bessel equation of order ν:

$$u'' + [k^2 - (4\nu^2 - 1)/4r^2]u = 0 \ . \tag{21}$$

This is obtained following a separation of variables which leads to a time dependence of the form $\exp(-k^2 t/4)$. The regular eigenfunctions are given by

$$u_k(r) \sim \sqrt{kr} \, J_\nu(kr) \ , \tag{22}$$

where J_ν is a Bessel function. Recalling the orthogonality relation for Bessel functions, the kernel of (19) can be written as

$$g(r,s;t) = \theta(t) \int_0^\infty dk \, k \, \sqrt{rs} e^{-k^2 t/4} \, J_\nu(kr) J_\nu(ks) \ . \tag{23}$$

This can be evaluated to give

$$g(r,s;t) = \theta(t) \, \frac{2}{t} \sqrt{rs} \exp\left(-\frac{r^2+s^2}{t}\right) I_\nu\left(\frac{2rs}{t}\right) \tag{24}$$

where I_ν is a modified modified Bessel function.

Finally, we equate the definition of $F(\rho,t)$ in (4) with its expression in terms of the kernel (24) as given by (10) and (18). Since this equality is valid for an arbitrary choice of the initial condition $\varphi(\sigma)$, the integrands of $d[S]$ must be the same.

By equating the coefficients of $\varphi(S)$ in the starting expression for $j(R)F(\rho,t)$ and $f(R,t)$ in (18) we arrive at

$$\int d\mu(U,V) \, \exp\left(-\frac{1}{t} \mathrm{tr}[(\sigma-\rho)^\dagger(\sigma-\rho)]\right)$$

$$= c \frac{t^{N_1 N_2 - N_2}}{\prod_{k=1}^{N_2} (r_k s_k)^\nu} \det_{k,l} \left| \exp\left(-\frac{r_k^2 + s_l^2}{t}\right) I_\nu\left(\frac{2 r_k s_l}{t}\right) \right| / \Delta(S^2)\Delta(R^2) \ . \tag{25}$$

Here, the constant c is determined by the normalization of the integrals.

This result enables us to calculate the Itzykson-Zuber integral for arbitrary complex matrices

$$\int d\mu(U,V)\, \exp\left(\mathrm{Re\,tr}\, U^\dagger S V R\right) = c\frac{1}{\prod_{k=1}^{N_2}(r_k s_k)^\nu} \det_{k,l} |I_\nu(r_k s_l)| \,/ \Delta(S^2)\Delta(R^2). \tag{26}$$

When $\nu = 0$, the product of factorials in this expression is understood to be 1. In the special case $N_1 = N_2$ for which $\nu = 0$, our expression reduces to the result of Guhr and Wettig [1] apart from a normalization constant.

12.2 Semiclassical Evaluation of the Integral

The final answer for the Itzykson-Zuber integral for rectangular matrices suggest that the integral is semiclassically exact. However this is not the case as can be easily seen from the case of 2×2 matrices. In that case the exponent in the Itzykson-Zuber integral is given by

$$\begin{aligned} &\mathrm{TrRe}\begin{pmatrix} e^{i\alpha}\cos\theta & -e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & e^{-i\alpha}\cos\theta \end{pmatrix}\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}\begin{pmatrix} e^{-i\alpha'}\cos\theta' & e^{-i\phi'}\sin\theta' \\ -e^{i\phi'}\sin\theta' & e^{i\alpha'}\cos\theta' \end{pmatrix}\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \\ = \ & r_1 s_1 \cos\theta\cos\theta'\cos(\alpha-\alpha') + (r_2 s_1 + r_1 s_2)\sin\theta\sin\theta'\cos(\phi-\phi') + r_2 s_2 \sin\theta\sin\theta'\cos(\alpha - \text{(27)} \end{aligned}$$

The integration measure is given by

$$2\sin\theta\cos\theta d\theta d\phi d\alpha 2\sin\theta'\cos\theta' d\theta' d\phi' d\alpha' \qquad . \tag{28}$$

It is clear that the integrand is not a total derivative and therefore the Duistermaat-Heckman theorem is not applicable.

12.3 Super Itzykson-Zuber integrals

Itzykson-Zuber integrals have also been derived for integrations over the super-unitary group [4, 5, 1]. In this section we only quote the result and refer the original literature for a derivation. The result is given by

$$\int dU e^{\mathrm{Trg} U S U^{-1} R} = \frac{\det e^{s_{1k} r_{1l}} \det e^{s_{2k} r_{2l}}}{B_k(s) B_k(r)}. \tag{29}$$

Here, s_{1k} and r_{1l} are the boson-boson diagonal elements of the diagonal matrices S and R, respectively. The nonzero fermion-fermion elements of the matrices S and R are denoted by is_{2k} and is_{2l}, respectively. The Berezinian is defined by

$$B_k(S) = \prod_{p<q} \frac{(s_{1p} - s_{1q})(is_{2p} - is_{2q})}{s_{1p} - is_{2q})(s_{1q} - is_{2p})} \prod_p \frac{1}{(s_{1p} - is_{2q})}, \tag{30}$$

and the same for $B_k(R)$. The Berezinian is the Jacobian for the change of variables

$$\sigma = U^{-1} S U \tag{31}$$

with

$$d\sigma = B_k^2(S) dS dU. \tag{32}$$

This result can be derived by means of a supersymmetric generalization of the diffusion equation.

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