13 The Calogero-Sutherland-Moser Model

In this chapter we explore the connection between exactly integrable systems knows as the Calogero-Sutherland-Moser models [1, 2, 3] and Random Matrix Theory. We consider the partition function

$$Z = \int d\phi(\tau) e^{-S(\phi(\tau))}, \tag{1}$$

where $\phi(\tau)$ are Hermitian matrices that depend on the continuous variable τ. They are either real, complex or quaternion real. The action is given by

$$S(\phi) = \int_{-T}^{T} \mathrm{Tr} \partial_\tau \phi \partial_\tau \phi + V(\phi). \tag{2}$$

Here, the potential $V(\phi)$ is an invariant function of ϕ. For $T \to \infty$ the partition function is dominated by the contribution from the ground state. The Hamiltonian corresponding to this partition function is given by

$$H = -\nabla_\phi^2 + V(\phi). \tag{3}$$

As usual we exploit the unitary invariance of the partition function. The matrices ϕ are diagonalized by

$$\phi = U \Lambda U^{-1}. \tag{4}$$

The integration measure in terms of the Λ and U is given by

$$d\Lambda dU \Delta^\beta(\Lambda). \tag{5}$$

It is completely standard to express the Laplacian in terms of the new coordinates

$$\sum_i \partial_{\phi_{ii}} + \sum_{i<j} \partial_{\mathrm{Re}\phi_{ij}} + \sum_{i<j} \partial_{\mathrm{Im}\phi_{ij}} = \frac{1}{\Delta^\beta(\Lambda)} \partial_{\lambda_i} \Delta^\beta(\Lambda) \partial_{\lambda_i} + f(\Lambda) \nabla_U^2. \tag{6}$$

WE expect that in the thermodynamic limit the ground state wave function does not depend on the angular variables, and is determined by the radial wave equation given by

$$-\frac{1}{\Delta^\beta(\Lambda)} \partial_{\lambda_i} \Delta^\beta(\Lambda) \partial_{\lambda_i} \psi + V(\lambda)\psi = E\psi. \tag{7}$$

This equation can be simplified by the introduction of a reduced wave function

$$\psi = \frac{F(\Lambda)}{\Delta(\Lambda)}, \tag{8}$$

resulting in the equation

$$-\partial_{\lambda_i}^2 F - F \frac{1}{\Delta^{\beta/2}(\Lambda)} \partial_{\lambda_i} \Delta^\beta(\Lambda) \partial_{\lambda_i} \frac{1}{\Delta^{\beta/2}(\Lambda)} + V(\Lambda) F = EF. \tag{9}$$

The derivative of the Vandermonde determinant is given by

$$\partial_{\lambda_i} \Delta^\beta(\Lambda) \partial_{\lambda_i} \frac{1}{\Delta^{\beta/2}(\Lambda)} = -\frac{\beta}{2}(\frac{\beta}{2} - 1) \Delta^{\beta/2-2} (\partial_{\lambda_i} \Delta)^2 - \frac{\beta}{2} \Delta^{\beta/2-1} \partial_{\lambda_i}^2 \Delta. \tag{10}$$

One easily shows that $\partial^2_{\lambda_i}\Delta = 0$. Another useful identity is that for three different eigenvalues

$$\frac{1}{\lambda_i-\lambda_k}\frac{1}{\lambda_i-\lambda_l}+\frac{1}{\lambda_k-\lambda_l}\frac{1}{\lambda_k-\lambda_i}+\frac{1}{\lambda_l-\lambda_i}\frac{1}{\lambda_l-\lambda_k}=0, \tag{11}$$

so that in the first term only the terms with two equal eigenvalues remains. This results in

$$-\partial^2_{\lambda_i}F+\frac{\beta}{2}(\frac{\beta}{2}-1)\sum_{i\neq k}\frac{1}{(\lambda_i-\lambda_k)^2}F+V(\Lambda)F=EF \tag{12}$$

This model is known as the Calogero model. For β different from 0 and 2, the behavior of the solution for $\lambda_i \to \lambda_k$ is given by

$$F \sim (\lambda_i-\lambda_k)^\alpha, \tag{13}$$

with $\alpha=\beta/2$ or $\alpha=1-\beta/2$. As boundary condition we choose the first possibility (also for $\beta=2$). For $\beta=2$ the solution is then a completely anti-symmetric function of the eigenvalues.

The Sutherland model can be obtained in the same way starting from unitary matrices instead of Hermitian matrices. Then the eigenvalues are located on the complex unit circle can be interpreted as the positions r_i of particles moving on the unit circle. The Sutherland Hamiltonian for N such particles is given by

$$H_S=-\sum_i\partial^2_{r_i}+\beta(\frac{\beta}{2}-1)(\frac{\pi}{N})^2\sum_{i<j}\frac{1}{\sin^2(\pi(r_i-r_j)/N)}. \tag{14}$$

13.1 Ground State of the Calogero-Sutherland-Moser Model

Generally our Schrödinger equation is given by

$$-\partial^2_{x_i}\psi+V\psi=E\psi, \tag{15}$$

where V is a pair potential

$$V=\sum_{i<j}v(x_i-x_j). \tag{16}$$

The N particle ground state of the ground state of such models can be obtained by introducing the Jastrow form given by

$$\psi=\prod_{i<j}\phi(x_i-x_j). \tag{17}$$

This results in the differential equation

$$-\sum_{i\neq k}\frac{\phi''(x_i-x_k)}{\phi(x_i-x_k)}-\sum_{i,k,l\ \text{different}}\frac{\phi'(x_i-x_k)\phi'(x_i-x_l)}{\phi(x_i-x_k)\phi(x_i-x_l)}+\sum_{i<j}v(x_i-x_j)=E. \tag{18}$$

A solution with a pair potential can only be obtained if the second term can be written as the sum of pairs. In the previous section we have seen that this is possible if $\phi(x)=x^\alpha$. Then the pair potential is equal to

$$U=\frac{2\alpha(\alpha-1)}{x^2}, \tag{19}$$

which is the Calogero model.
Exercise. Show that the Sutherland model is obtained for $\phi = \sin^\lambda x$.

A generalized Jastrow Ansatz is given by

$$e^{-\sum_k V(x_k)} \prod_{k<l} \phi(x_k - x_l). \tag{20}$$

A Gaussian potential, $V(x) = x^2/2$, and $\phi(x) = x^\alpha$ is the solution of the pair potential

$$\begin{aligned} V &= \alpha(\alpha-1)\sum_{i\neq j}\frac{1}{(x_i-x_j)^2} + \sum_j (x_j^2 - 1) + \alpha \sum_{j\neq l}\left[\frac{-x_j}{x_j - x_l} + \frac{-x_l}{x_l - x_j}\right] \\ &= \sum_j (x_j^2-1) + 2\alpha(\alpha-1)\sum_{j<l}\frac{1}{(x_j-x_l)^2}. \end{aligned} \tag{21}$$

For $\alpha = \frac{1}{2}$, 1, or 2, the square of the wave function is equal to the joint probability density of the Gaussian random matrix ensemble with $\beta = 2\alpha$. The pair correlation function then follows from the two-point correlation function of two eigenvalues.

13.2 Time Dependent Correlation Function

In this section we show that time dependent correlation functions in the CSM model can be calculated by means of the supersymmetric method of Random Matrix Theory. We will study the correlation function

$$\int d\phi(\tau) \mathrm{Tr}\frac{1}{z_1 - \phi(t_1)} \mathrm{Tr}\frac{1}{z_2 - \phi(t_2)} e^{-S(\phi(\tau))}, \tag{22}$$

where z_1 has a positive imaginary increment and z_2 has a negative imaginary increment. The generating function for this correlation function is given by

$$\int d\phi(\tau) d\psi \exp\left[\begin{pmatrix}\phi_1^* \\ \phi_2^* \\ \chi_1^* \\ \chi_2^*\end{pmatrix}\begin{pmatrix} z_1 - \phi(t_1) & 0 & 0 & 0 \\ 0 & -z_2 + \phi(t_2) & 0 & 0 \\ 0 & 0 & z_1 + J - \phi(t_1) & 0 \\ 0 & 0 & 0 & z_2 + J - \phi(t_2)\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2 \\ \chi_1 \\ \chi_2\end{pmatrix}\right] e^{-S(\phi(\tau))}, \tag{23}$$

where

$$d\psi = d\phi_1 d\phi_2 d\chi_1 d\chi_2 d\phi_1^* d\phi_2^* d\chi_1^* d\chi_2^*. \tag{24}$$

The action can be diagonalized by Fourier transforming the matrix field

$$\phi(t) = \frac{1}{\sqrt{\pi}} \int d\omega e^{i\omega t} \phi(\omega). \tag{25}$$

Because $\phi(t)$ is Hermitian we have that

$$\phi(\omega)^\dagger = \phi(-\omega). \tag{26}$$

The action of the Fourier transformed variables is given by

$$S(\phi(\omega)) = \int d\omega (\omega^2 + \omega_0^2) \sum_{kl} \phi_{kl}(\omega)\phi_{kl}^*(\omega), \tag{27}$$

where we have chosen a Gaussian potential $V(x) = \omega_0^2 x^2$. The remaining ϕ dependence resides in

$$\begin{aligned} & e^{-i\phi_{ij}(t_1)(\phi_1^{i\,*}\phi_1^j+\chi_1^{i\,*}\chi_1^j)}e^{-i\phi_{ij}(t_1)(-\phi_2^{i\,*}\phi_2^j+\chi_2^{i\,*}\chi_2^j)} \\ = \; & e^{-2i\int d\omega\phi_{ij}(\omega)(e^{i\omega t_1}A_{ij}+e^{i\omega t_2}B_{ij})}. \end{aligned} \tag{28}$$

By using the Hermiticity property of $\phi(\omega)$ this expression can be rewritten as

$$e^{-i\int d\omega[\phi_{ij}(\omega)(e^{i\omega t_1}A_{ij}+e^{i\omega t_2}B_{ij})+\phi_{ij}^*(\omega)(e^{-i\omega t_1}A_{ji}+e^{-i\omega t_2}B_{ji})]}. \tag{29}$$

The integral over $\phi(\omega)$ can be performed by simply completing the square and perform the Gaussian integral. The generating function is thus given by

$$\begin{aligned} Z \; = \; & \int d\psi e^{-\int \frac{d\omega}{\omega_0^2+\omega^2}(\mathrm{Tr}A^2+\mathrm{Tr}B^2+(e^{i\omega(t_1-t_2)}+e^{-i\omega(t_1-t_2)})\mathrm{Tr}AB)} \\ \times \; & e^{iz_1(\phi_1^*\phi_1+\chi_1^*\chi_1)+iz_2(-\phi_2^*\phi_2+\chi_2^*\chi_2)+J(\chi_1^*\chi_1+\chi_2^*\chi_2)}. \end{aligned} \tag{30}$$

All terms in this equation appear as singlets with respect to the unitary invariance of the original matrix model. After decoupling the quartic terms via a Hubbard-Stratonovitch transformation, the integral of the ψ variables can be performed and we end up with an integral over 4 Grassmann variables. This integral can be performed explicitly, or alternatively, we can further simplify these expressions by using the eigenvalues and eigenvectors of the super matrices as new integration variables. For the of this calculation we refer to the original literature [4, 5].

References

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