14 Hilbert Space Quantization of a Random Hopping Model

There are two different ways of doing field theory. First, by means of path integrals where the fields are represented by complex functions or Grassmann valued functions. Second, in a second quantized formalism where the fields are operators. In the first case one usually works with imaginary time resulting in convergent integrals. I emphasize that convergence is essential for truly nonperturbative studies. In the second case one usually uses Minkowsky space time, and constructs a Hilbert space that provides the particle content of the fields.

In this chapter we will go from a path integral formulation to a Hilbert space formulation by means of coherent state path integrals. This allows us to derive the RMT result for the GUE two-point correlation function from a sum over all zero energy states. We will study this problem for a 1-dimensional hopping model with off-diagonal disorder. This model [1] is important for the study of edge states in quantum Hall effect. Below we follow the discussion of this topic as given in ref. [1].

14.1 Coherent State Path Integrals

We will discuss two type of coherent states, bosonic coherent states and fermionic coherent states. Bosonic coherent states are defined by

$$|\phi>= e^{\sum_\alpha \phi_\alpha^{\dagger\alpha}}|0>, \tag{1}$$

where the $a^\dagger_\alpha$ are the usual bosonic creation operators satisfying the commutation relations

$$[a_\alpha, a^\dagger_\beta] = \delta_{\alpha\beta}. \tag{2}$$

Coherent states are eigenstates of the annihilation operator

$$a_\alpha|\phi>= \phi_\alpha|\phi> . \tag{3}$$

The conjugate coherent states are defined by

$$<\phi| =< 0|e^{\sum_\alpha \phi^*_\alpha a_\alpha}, \tag{4}$$

and satisfy the relation

$$<\phi|a^\dagger_\alpha =<\phi|\phi^*_\alpha. \tag{5}$$

Coherent states are an overcomplete set normalized according to

$$<\phi|\psi>= e^{\sum_\alpha \phi^*_\alpha\psi_\alpha}. \tag{6}$$

The resolution of the identity is given by

$$1 = \int \prod_\alpha \frac{d\phi d\phi^*}{2\pi i}|\phi><\phi|e^{-\sum_\alpha \phi^*_\alpha\phi_\alpha}. \tag{7}$$

Exercise 14.1 Prove these relations.

Another important relation is that the coherent state expectation value of a normal ordered operator is given by

$$< \xi|A(a_\alpha^\dagger a_\alpha|\xi' >= e^{\sum_\alpha \xi_\alpha^* \xi_\alpha{}'} A(\xi_\alpha, \xi_\alpha{}'). \tag{8}$$

Fermionic coherent states are defined a similar way,

$$|\psi > e^{-\sum \psi_\alpha b_\alpha^\dagger}|0> \tag{9}$$

where the $b_\alpha^\dagger$ are fermionic creation operators satisfying the anti-commutation relations

$$\{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta}. \tag{10}$$

In this case the ψ_α are Grassmann valued variables. The fermionic coherent states are also eigenstates of the annihilation operator,

$$b_\alpha|\psi >= \xi_\alpha|\psi > . \tag{11}$$

The conjugate coherent states defined by

$$< \xi| =< 0|e^{-\sum_\alpha b_\alpha \xi_\alpha^*} \tag{12}$$

are eigenstates of the creation operator

$$< \xi|b_\alpha^\dagger =< \xi|\xi_\alpha^*. \tag{13}$$

In this case the resolution of unity is given by

$$1 = \int \prod_\alpha d\xi_\alpha d\xi_\alpha^* e^{-\sum_\alpha \xi_\alpha^* \xi_\alpha}|\xi >< \xi|, \tag{14}$$

and two fermionic coherent states are normalized according to

$$\langle \xi|\xi' \rangle = e^{\sum_\alpha \xi_\alpha^* \xi_\alpha}. \tag{15}$$

A coherent state path integral can be derived in the usual way by splitting the time evolution operator in small time intervals and repeatedly inserting the resolution of unity. One easily derives that

$$< \phi_f| : e^{-\beta H} : |\phi_i >= \int_{\phi_\alpha(t_i)=\phi_{i\alpha}}^{\phi_\alpha(t_f)=\phi_{f\alpha}} D\phi_\alpha D\phi_\alpha^* e^{\sum_\alpha \phi_\alpha^*(t_f)\phi_\alpha(t_f)} e^{\int_{t_i}^{t_f} d\tau(-H(\phi,\phi^*)-\sum_\alpha \phi_\alpha^* \partial_\tau \phi_\alpha)} \tag{16}$$

The partition function is the trace over all states and can therefore also be written as an integral over coherent states

$$\begin{aligned} Z &= \mathrm{Tr} e^{-\beta H} \int d\phi d\phi^* e^{-\sum_\alpha \phi_\alpha^* \phi_\alpha} < \phi|e^{-\beta H}|\phi > \\ &= \int_{\phi(\beta)=\phi(0)} D\phi D\phi^* e^{-\int_0^\beta d\tau(\phi^* \partial_\tau \phi + H)}, \end{aligned} \tag{17}$$

where the Hamiltonian in the left hand side is normal ordered. The only difference for the partition function with fermionic coherent states is that the integrals are over Grassmann fields with anti-periodic boundary conditions. For a derivation of these results and additional details we refer to the literature [2]. The above summary is based on this book as well.

14.2 The model

We consider a Hamiltonian given by

$$H = \sum_{n=1}^{N} \int_0^\beta dx [\psi_n^\dagger i \partial_x \psi_n - (t_n \psi_n^\dagger \psi_{n+1} + t_n^* \psi_{n+1}^\dagger \psi_n). \tag{18}$$

The second cumulant of the Gaussian random hopping matrix elements is given by

$$< t_n(\tau) t_m(\tau') >= D \delta_{nm} \delta(\tau - \tau'). \tag{19}$$

The generating function of the one-particle Greens functions of this partition function is given by

$$Z = \int d\psi d\bar{\psi} d\phi d\phi^* \langle e^{-\sum_n \int dx \mathcal{L}_n} \rangle, \tag{20}$$

where the average $\langle \cdots \rangle$ is over the hopping matrix elements. The Lagrangian $\mathcal{L}_n$ is given by the sum of a bosonic and a fermionic piece

$$\mathcal{L}_n = \phi_n^* \sigma_z \partial_x \phi_n + i(t_n \phi_n^* \sigma_z \phi_{n+1} + c.c.) + \eta \phi_n^* \phi_n + (\phi \to \psi). \tag{21}$$

The fermionic part of the Lagrangian is obtained by replacing ϕ^* and ϕ by $\bar{\psi}$ and ψ. All fields have two components corresponding to the advanced an retarded Greens functions. The Pauli matrix σ_z arises because of the requirement of having convergent integrals. The convergence is determined by the imaginary increment of η which is opposite for the retarded and the advanced Greens functions.

The Greens functions are given by

$$G_+(n, n'; x, x'; \omega/2) = \langle nx|(H + i\eta - \omega/2)^{-1}|n'x'\rangle = -i < \psi_{n1}(x) \psi_{n'1}^*(x') > \tag{22}$$
$$G_-(n, n'; x, x'; \omega/2) = \langle nx|(H - i\eta + \omega/2)^{-1}|n'x'\rangle = i < \psi_{n2}(x) \psi_{n'2}^*(x') >, \tag{23}$$

where the expectation value is with respect to the generating function.

The average over the randomness can be performed trivially by means of a cumulant expansion. Averaging over the hopping matrix element results in the factor

$$e^{-\frac{D}{2} \sum_n (A_{n+1,n} A_{n,n+1} + A_{n,n+1} A_{n+1,n})} \tag{24}$$

where

$$A_{n,m} = \phi_{n1}^* \phi_{m1} - \phi_{n2}^* \phi_{m2} + (\phi \to \psi). \tag{25}$$

Next we wish to convert the generating function into a coherent state path integral. If we interpret the x coordinate as the imaginary time the partition function has exactly the form of a coherent state path integral except for the sign of the 2-terms. However this can be easily corrected by interchanging ϕ_{n2} and ϕ_{n2}^* in the bosonic case and

$$\psi_{n2} \quad \to \quad -\bar{\psi}_{n2}, \tag{26}$$
$$\bar{\psi}_{n2} \quad \to \quad \psi_{n2} \tag{27}$$

for the fermionic variables.

Now we are in a position to write down the Hamiltonian of our system,

$$H = H_D + H_\eta, \tag{28}$$

with H_D given by

$$H_D = \frac{D}{2}\sum_n (A_{n+1\,n}A_{n\,n+1} + A_{n\,n+1}A_{n+1\,n}), \tag{29}$$

and

$$H_\eta = \eta\sum_n (b_n^\dagger b_n + f_n^\dagger f_n). \tag{30}$$

The operator A_{nm} is given by

$$A_{nm} = b_{n1}^\dagger b_{m1} - b_{n2}^\dagger b_{m2} + (b \to f). \tag{31}$$

In absence of a super-symmetry breaking source term, the fermionic and the bosonic integrals should cancel exactly. This implies that the boundary condition for the fermion fields and the boson fields are the same. However, in a path integral, the fermion fields should satisfy anti-periodic boundary condition, $\psi(0) = -\psi(\beta)$. This can be achieved by making a gauge transformation

$$\psi_n(\tau) = e^{i\pi\tau/\beta}\psi_n(\tau). \tag{32}$$

The kinetic term in the Lagrangian is not invariant under this transformation resulting in the extra contribution

$$\frac{\pi i}{\beta}\sum_n \bar{\psi}_n\sigma_z\psi_n. \tag{33}$$

At the operator level this results in the extra term in the Hamiltonian

$$H_{bc} = \frac{\pi i}{\beta}\sum_n f_n^\dagger f_n. \tag{34}$$

Since the total number of fermion is a good quantum number of our Hamiltonian, this term results in the factor $(-1)^{n_f}$ in the expectation value of an operator

$$\langle O\rangle = \mathrm{Str}Oe^{-\beta(H+H_{bc})} = \mathrm{Str}O(-1)^{n_f}e^{-\beta H}. \tag{35}$$

14.3 Random Matrix Theory limit

In this section we derive the conditions for which the partition function can be reduced to a zero dimensional theory with eigenvalue correlations given by RMT.

The randomness in the hopping matrix elements give rise to diffusive motion in the space of edge states. However, the motion in the τ direction in unimpaired. We can thus distinguish two different scales, corresponding to the times it takes to traverse the τ spaces and the time it takes to traverse the edge states. The first time scale is given by β. The find the second time scale we note that the disorder results in diffusive motion in the edge states with $\delta n = \sqrt{D\delta\tau}$ (the diffusion constant is denoted by D). The time it takes to diffuse over all states is then given by

$$\tau_D = \frac{N^2}{D}. \tag{36}$$

We expect that Random Matrix Theory is valid on time scales where an initial localized wave diffuses all over over space, i.e. for times larger than the diffusion time τ_D. The partition function is given by the trace of the imaginary time evolution operator for time β. So, we expect Random Matrix Theory for

$$\beta \gg \tau_D. \tag{37}$$

This condition can be rewritten as

$$\beta D \gg N^2. \tag{38}$$

The Hamiltonian H_D is a discretized Laplacian, and the lowest nonzero energy eigenvalue is thus given by

$$E_{\rm min} \sim D(\frac{2\pi}{N})^2. \tag{39}$$

In the partition function these states contribute as

$$e^{-\beta E_{\rm min}} = e^{-\beta D(2\pi/N)^2}. \tag{40}$$

In the domain (38) the contribution of these states and all higher excited states can be neglected with respect to the zero energy states, and the partition function can be approximated by the trace over the zero energy states of H_D. Because of the presence of H_η, which does not vanish for the zero energy states of H_D, we obtain nontrivial results.

14.4 The Zero Energy States of H_D

Obviously, the state with zero fermions and zero bosons has zero energy. However, this state is highly degenerate. In this section we discuss the symmetries of H_D which will enable us to construct all zero energy states.

The operators A_{nm} do not change the total number of 1- and 2- bosons and fermions. Therefore, the operators

$$n_{b1},\ n_{b2},\ n_{f1},\ n_{f2}, \tag{41}$$

commute with H_D. Because A_{nm} is invariant for the interchange of either 1- or 2- bosons and fermions, a second set of operators that commutes with H_D is given by

$$\begin{aligned} Q_i &= \sum_n b^\dagger_{ni} f_{ni}, \quad i = 1,2, \\ \bar{Q}_i &= \sum_n f^\dagger_{ni} b_{ni}, \quad i = 1,2. \end{aligned} \tag{42}$$

The last group of symmetries is given by adding or subtracting a 12-pair of bosons or fermions. Specifically, the following operators

$$\begin{aligned} S^+ &= \sum_n f^\dagger_{n1} f^\dagger_{n2}, &(43)\\ J^+ &= \sum_n b^\dagger_{n1} b^\dagger_{n2}, &(44)\\ A^+ &= \sum_n f^\dagger_{n1} b^\dagger_{n2}, &(45)\\ \tilde{A}^+ &= \sum_n b^\dagger_{n1} f^\dagger_{n2}, &(46)\end{aligned}$$

and the operators obtained by replacing the creation operators by annihilation operators commute with H_D.

Let us show explicitly that S^+ commutes with A_{nm}. Obviously, S^+ commutes with the bosonic piece of A_{nm}. Therefore, we only have to calculate the commutator

$$\begin{aligned} & [f^\dagger_{m1} f_{n1} - f^\dagger_{n2} f_{m2}, \sum_k f^\dagger_{k1} f^\dagger_{k2}] \\ = & f^\dagger_{m1}(-f^\dagger_{k1} f_{n1} + \delta_{kn}) f^\dagger_{k2} - f^\dagger_{k1} f^\dagger_{n2}(-f^\dagger_{k2} f_{m2} + \delta_{km}) \\ = & 0. \end{aligned} \tag{47}$$

Exercise. Show by an explicit calculation that also the other 15 operators commute with A_{nm}.

One can also easily show that the operators S^+, J^+, A^+ and $\tilde{A}^+$ commute among themselves. The other 12 operators all contain annihilation operators. Therefore, all zero energy states can be generated from acting with S^+, J^+, A^+ and $\tilde{A}^+$ on the vacuum state. The operators A^+ and $\tilde{A}^+$ are anti-commuting operators. We thus have

$$(A^+)^2 = 0, \qquad (\tilde{A}^+)^2 = 0. \tag{48}$$

The complete set of zero energy states is therefore given by

$$\begin{aligned} |n_1 n_2 00> &= (S^+)^{n_1} (J^+)^{n_2} |0>, &(49)\\ |n_1 n_2 10> &= (S^+)^{n_1} (J^+)^{n_2 -1} A^+ |0>, &(50)\\ |n_1 n_2 01> &= (S^+)^{n_1} (J^+)^{n_2 -1} \tilde{A}^+ |0>, &(51)\\ |n_1 n_2 11> &= (S^+)^{n_1} (J^+)^{n_2 -2} A^+ \tilde{A}^+ |0> . &(52)\end{aligned}$$

Because of the Pauli principle, n_1 ranges from 0 till $N-1$ for the second and third set of states. In the states $|N n_2 00>$ and $|N-1 n_2 - 111>$ all fermionic states are filled, and therefore represent the same fermionic states. In order to avoid double counting, we also can take n_1 in the range $[0, N-1]$ in the first set of states. For $n_2 = 1$ the last set of states is not defined. However, in the first set of states we can take $n_2 = 0$ which gives us a extra set of states. If we replace $|n_1 111>$ by $|n_1 000>$ we find that our quantum numbers are in the range

$$\begin{aligned} 0 \leq \ & n_1 \leq N-1, &(53)\\ 1 \leq \ & n_2 < \infty. &(54)\end{aligned}$$

With the above exceptions, each state with given n_1 and n_2 occurs as a super-symmetric quadruplet. The only exception is the vacuum states with zero bosons and zero fermions. The states can be classified more elegantly in terms of the lowest weight states of the Q operators (see ref. [1]).

Nontrivial results are obtained because H_η lifts the degeneracy of the zero energy states. One immediately finds that

$$H_\eta|n_1n_2\mu\nu>=2\eta|n_1n_2\mu\nu> . \tag{55}$$

The calculation of the spectral correlation functions is thus reduced to a sum over zero energy states with $\exp -\beta H_\eta$ as a weight factor.

14.5 The Two-Point Correlation Function

In this section we calculate spectral correlation function in the domain where we can restrict ourselves to the zero energy states.

The spectral density is given by

$$\rho(0) = \int_0^\beta d\tau \sum_{n=1}^N \frac{1}{2\pi}(G_- - G_+)(n,n,\tau,\tau). \tag{56}$$

In terms of an expectation value with respect to the generating function the Greens functions are given by

$$G_\alpha(n,n',\tau,\tau') = -\alpha i < \psi_{n\alpha}(\tau)\bar{\psi}_{n'\alpha}(\tau') > . \tag{57}$$

To transcribe this expectation value into the expectation value of an operator we have to pay special attention to operators at equal time. We use the standard prescription that

$$\psi_n\bar{\psi}_n \to \frac{1}{2}(f_nf_n^\dagger - f_n^\dagger f_n) = f_nf_n^\dagger - \frac{1}{2}. \tag{58}$$

We then find that in the spectral density can be expressed as

$$\rho(0) \quad = \quad \frac{\beta}{2\pi}\mathrm{Tr}(n_{f1}+n_{f2}+N)(-1)^{n_{f1}+n_{f2}}e^{-\beta H_\eta}.$$

where the term N results from the term -1 in (58). Because of the factor $(-1)^{n_{f1}+n_{f2}}$ the contribution of all nonzero energy multiplets to this state vanish. The only remaining contribution is the contribution of the vacuum state. The contribution of the other terms vanishes for each supersymmetric quadruplet because $2n_1-(2n_1+1)-(2n_1+1)+2n_1+2=0$ for the four states in the supersymmetric multiplet. In that case also the vacuum contribution vanishes. One can easily verify that there are no problems with the exceptions mentioned in the previous section. We finally find that

$$\rho(0) = \frac{N\beta}{2\pi}. \tag{59}$$

The two-point spectral correlation function is given by

$$C(\omega) = \langle\frac{\beta^2}{4\pi^2}\sum_{nn'}(G_{n1}(\omega/2)G_{n'2}(-\omega/2)+G_{n1}(-\omega/2)G_{n'2}(\omega/2)]\rangle_c \tag{60}$$

where $G_{ni}(\omega)$ is the Greens function

$$G_{ni}(\omega) = G_i(n, n, \tau, \tau, \omega). \tag{61}$$

The average $\langle \cdots \rangle_c$ is the connected average with the product of the average over the Greens functions subtracted. In the operator formalism this amounts to excluding terms with $n_1 = n_2 = 0$. The correlation function is thus given by

$$\begin{aligned} C(\omega) &= \frac{\beta^2}{4\pi^2}\mathrm{Tr}'\left[\sum_{nn'}[(f^\dagger_{n1}f_{n1}-\frac{1}{2})(f^\dagger_{n'2}f_{n'2}-\frac{1}{2})e^{-\beta H_\eta}+(f^\dagger_{n2}f_{n2}-\frac{1}{2})(f^\dagger_{n'1}f_{n'1}-\frac{1}{2})e^{-\beta H_{\eta^*}}](-1)^{n_f}\right] \\ &= \frac{\beta^2}{4\pi^2}\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{\infty}(e^{-\beta 2\eta(n_1+n_2)}+e^{-\beta 2\eta^*(n_1+n_2)}) \end{aligned} \tag{62}$$

where n_f is the fermion number of the state, and Tr' indicates that we exclude the vacuum state. It is straightforward to perform the sum over the zero energy states. With the $n_1 = n_2 = 0$ state excluded, the $\frac{1}{2}$ terms sum to zero. For each supersymmetric multiplet the pre-exponential factor sums to

$$n_1^2 - 2n_1(n_1+1) + (n_1+1)^2 = 1. \tag{63}$$

For large N the sum over n_1 and n_2 in (62) can be approximated by an integral. With $x = n_2/N$ and $y = n_1/N$ we obtain

$$C(\omega) = \frac{N^2\beta^2}{2\pi^2}\mathrm{Re}\int_0^\infty dx\int_0^1 dy e^{-\beta N(2\eta+i\omega)(x+y)}. \tag{64}$$

where we have replaced $\eta \to \eta + \omega/2$. The integral are elementary. Taking the limit $\eta \to 0$ after the integration we find

$$C(\omega) = \frac{1}{2\pi^2}\frac{\cos N\beta\omega - 1}{\omega^2}, \tag{65}$$

$$= -\frac{\sin^2(N\beta\omega/2)}{\pi^2\omega^2}. \tag{66}$$

If we express the energy difference in units of the average level spacing

$$r = \omega\rho(0) = \frac{N\beta\omega}{2\pi}, \tag{67}$$

we find for the normalized correlation function

$$\frac{C(\omega)}{\rho^2(0)} = -\frac{\sin^2(\pi r)}{\pi^2 r^2}, \tag{68}$$

which is the correct result for the Gaussian Unitary Ensemble.

References

[1] L. Balents, M.P.A. Fisher and M.R. Zirnbauer, cond-mat/9608049.

[2] J. Negele and H. Orland, *Quantum Many Particle Systems*, Addison Wesley Publishing Co., 1988.