On the caculation of some integrals

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1 Introduction

In these lectures we discuss the calculation of some integrals that occur in the theory of random matrices. Random Matrix Theory (RMT) was first introduced in physics by Wigner to obtain a description of the statistical properties of level spacings in nuclei [1]. Later Random Matrix Theory has been applied successfully to a wide spectrum of fields. Among others we mention the spacings of zeros of the Riemann ζ-function, the correlations of resonances in quartz blocks and the theory of exactly soluble systems. For an extensive overview of the various applications of RMT we refer to the recent review by the Heidelberg group [2]. The true classic for the classical methods of Random Matrix Theory is the book by Mehta [3].

Random Matrix Theory is the study of Ensembles of Random matrices with a given probability distribution. We will study probability distributions that are invariant with respect to cerain symmetry groups. We will consider three types of ensembles of invariant Hermitian Random Matrices. Ensembles with real elements ($\beta = 1$), ensembles with complex ($\beta = 2$) elements, and ensembles with quaternion real elements ($\beta = 4$). They are invariant under the transformation

$$H \to UHU^{-1}, \tag{1}$$

where U is either orthogonal, unitary or symplectic for $\beta = 1$, 2, or 4, respectively.

If, in addition to the invariance of the random matrix ensemble, we require the statistical independence of the matrix elements, it can be shown that the probability distribution is necessarily Gaussian (see [4]).
Exercise. Prove this.

In general the probability distribution is not Gaussian. Except for the invariance and some convergence requirement we will make no further assumptions for the probability distribution. For $\beta = 1$, 2 or 4, the probability distribution for the eigenvalues is then given by

$$P(\{x\})d\{x\} = \mathcal{N}|\Delta(\{x\})|^\beta \prod_k w(x_k)dx_k, \tag{2}$$

where the Vandermonde determinant is defined by

$$\Delta(\{x\}) = \prod_{k>l}(x_k - x_l) \tag{3}$$

The Vandermonde determinant arises through the Jacobian from the transformation of the matrix elements as integration variables to the eigenvalues and eigenvector as integration variables. The distribution of the eigenvectors completely factorizes from the probability distribution. The functions $w(x)$ are specific for the probability distribution. In the simplest case, with $w(x)$ a

Gaussian function, the corresponding Random Matrix Ensembles will be called the Gaussian Orthogonal Ensemble, the Gaussian Unitary Ensemble and for Gaussian Symplectic Ensemble, for $\beta = 1$, $\beta = 2$ and $\beta = 4$, respectively. .

The definition of eigenvalues is ambiguous for quaternion real matrices as can be concluded form the following theorem. If we define a determinant according to the conditions:

- $\det A = 0$, if and only if we can find a nonzero solution to the equation $Ax = 0$,
- $\det AB = \det A \det B$,
- $\det A$ is a linear function of the rows of A,

the multiplication law of our number field is necessarily commutative [7]. This invalidates the usual definition of a determinant in case of quaternion real matrix elements. Instead we introduce the definition

$$\det A = \sum_P \mathrm{sign}(P) \prod_{\text{cycles}} (A_{i_1 i_2} A_{i_2 i_3} \cdots)_0 (A_{i_p i_{p+1}} \cdots)_0 \cdots, \tag{4}$$

where the subscript $(\cdots)_0$ denotes the scalar part of the quaternion.
Exercise. Show that the subscripts zero in eq. (4) can be removed if A is selfdual and a certain order in maintained in the product of the cycles [5].

For selfdual A, it can be shown that this definition satisfies the equality

$$\det A = \mathrm{Pf}(ZC(A)). \tag{5}$$

Here, $Z = C(E_1)$ with E_1 the quaternion matrix with e_1 on the diagonals. For a self-dual quaternion matrix, the matrix $ZC(A)$ is anti-symmetric, and the Pfaffian is well-defined. For completeness, we also mention that if $ZC(A)$ is anti-symmetric, the quaternion matrix A is self-dual. This definition of the determinant will be useful later.

Below we will calculate the correlation functions of the eigenvalues. They are given by the integrals

$$I_{\beta m} = \int |\Delta(\{x\})|^\beta \prod_{k=1}^{n} w(x_k) dx_{m+1} \cdots dx_n, \tag{6}$$

2 Integration over all eigenvalues

2.1 $\beta = 2$

We first consider the case $\beta = 2$. This case is always much simpler than the cases $\beta = 1$ or $\beta = 4$. We start by rewriting the Vandermonde determinant as

$$\Delta(\{x_k\}) = \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_N \\ \vdots & & \vdots \\ x_1^{N-1} & \cdots & x_N^{N-1} \end{vmatrix}. \tag{7}$$

By the addition of multiples of rows of lower order which do not change the determinant, the Vandermonde determinant can be rewritten as

$$\Delta(\{x_k\}) = \begin{vmatrix} P_0(x_1) & \cdots & P_0(x_N) \\ P_1(x_1) & \cdots & P_1(x_N) \\ \vdots & & \vdots \\ P_{N-1}(x_1) & \cdots & P_{N-1}(x_N) \end{vmatrix}, \tag{8}$$

where the $P_k(x)$ are monic polynomials of order k. The square of the Vandermonde determinant can thus be written as

$$\Delta^2(\{x_k\}) = \det P_{j-1}(x_i) \det Q_{j-1}(x_i), \tag{9}$$

with $P_k(x)$ and $Q_l(y)$ monic polynomials with order given by their index. Our integral is then given by

$$\begin{aligned} I_{2\,0} &= \int dx_1 \cdots dx_n \sum_{\sigma\rho} (-1)^\sigma (-1)^\rho \prod_k w(x_k) P_{\sigma(1)}(x_1) \cdots P_{\sigma(n)}(x_n) P_{\rho(1)}(x_1) \cdots P_{\rho(n)}(x_n), \\ &= \sum_{\sigma\rho} (-1)^\sigma (-1)^\rho G_{\sigma(1)\rho(1)} \cdots G_{\sigma(1n)\rho(n)} \\ &= n! \det G, \end{aligned} \tag{10}$$

where the matrix G is given by

$$G_{jk} = \int dx w(x) P_j(x) Q_k(x). \tag{11}$$

As an example we mention the calculation of the probability of finding no eigenvalues inside the interval I. In that case we modify the weight function according to (with $\chi_I(x)$ is the characteristic function of the interval I)

$$w(x) \to w(x)(1 - z\chi_I(x)), \tag{12}$$

and calculate the integral over all eigenvalues. By differentiating this result k times with respect to z we obtain the probability of finding exactly k eigenvalues inside this interval.

2.2 $\beta = 4$

Next we consider the case $\beta = 4$. We use the property that the fourth power of the Vandermonde determinant can be written as

$$\Delta^4(\{x\}) = \begin{vmatrix} 1 & 0 & \cdots & 1 & 0 \\ x_1 & 1 & \cdots & x_n & 1 \\ x_1^2 & 2x_1 & \cdots & x_n^2 & 2x_n \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^{2n-1} & (2n-1)x_1^{2n-2} & \cdots & x_n^{2n-1} & (2n-1)x_n^{2n-2} \end{vmatrix}. \tag{13}$$

By the addition and subtraction of rows, this determinant can be rewritten as

$$\Delta^4(\{x\}) = \det |P_{j-1}(x_i), {P'}_{j-1}(x_i)|, \tag{14}$$

where $P_{j-1}(x)$ is an arbitrary monic polynomial of order $j-1$ and its derivative is denoted by a prime. The index j runs from 1 till $2n$ and the index i from 1 till n. If we define the overlap matrix elements

$$G_{jk} = -G_{kj} = \int dx w(x)(P_j(x)P'_k(x) - P'_j(x)P_k(x)), \tag{15}$$

the integral over all eigenvalues for $\beta = 4$ can be expressed as

$$\begin{aligned} I_{40} &= \int \Delta^4(\{x\})dx_1 \cdots dx_n \\ &= \sum_\sigma \mathrm{sign}(\sigma) G_{\sigma_1\sigma_2} G_{\sigma_2\sigma_3} \cdots G_{\sigma_{2n-1}\sigma_{2n-2}} \\ &= n!\mathrm{Pf}(G). \end{aligned} \tag{16}$$

This result follows from expanding the determinant as a sum over permutations and then reordering the terms. By definition the result is equal to the Pfaffian of G, which, up to a sign is equal to the square root of the determinant of G.

It is natural to introduce a scalar product by

$$< f, g >_4 = \int (f(x)g'(x) - f'(x)g(x))w(x)dx, \tag{17}$$

and the polynomials that are orthogonal with respect to this scalar product,

$$< P_{2j}, P_{2k+1} >_4 = - < P_{2k+1}, P_{2j} >_4 = \delta_{jk}, \tag{18}$$

$$< P_{2j}, P_{2k} >_4 = < P_{2k+1}, P_{2j+1} >_4 = 0. \tag{19}$$

These polynomials are known as the skew-orthogonal polynomials of the first kind. In general, these polynomials are not monic, and (16) has to be modified to take into account the normalization constant.

2.3 $\beta = 1$

Finally, we consider $\beta = 1$. In this case the integral is given by

$$I_{10} = \int |\Delta(\{x\})| \prod w(x_i) dx_1 \cdots dx_n. \tag{20}$$

The integrand is a symmetric function of the x_k, which allows us to rewrite the integral as

$$I_{10} = n! \int_{x_1 \le x_2 \le \cdots \le x_n} \Delta(\{x\}) \prod w(x_i) dx_1 \cdots dx_n, \tag{21}$$

where the absolute value sign has been dropped. By the addition of columns, the Vandermonde determinant can be expressed in terms of arbitrary monic polynomials as,

$$\Delta(\{x\}) = \det |P_{j-1}(x_i)|. \tag{22}$$

Next, we perform the integrations over the x_k with an odd index inside the determinant. The j'th row of the argument of the determinant is then given by

$$\int_{-\infty}^{x_2} w(x_1)p_j(x_1)dx_1, P_j(x_2), \int_{x_2}^{x_4} w(x_3)P_j(x_3)dx_3, \cdots \tag{23}$$

For simplicity we take n even. However, the present calculation can be extended with only minor complications to odd n. By the addition of the first column to the third column, the lower limit of the integrals over x_3 can be extended to $-\infty$. Similarly, one can extend the lower limit of all integrals inside the determinant to $-\infty$. Our integral is then given by

$$I_{10} = n! \int_{x_2 \le x_4 \le \cdots x_n} dx_2 dx_4 \cdots dx_n \det | \int_{-\infty}^{x_2} P_j(x_1) dx_1, P_j(x_2), \cdots |. \tag{24}$$

Because the x_k with even k occur in two rows of the argument of the determinant, the determinant is a symmetric function of the x_{2i}. Therefore, the restriction on the integrations over the x_{2i} can be eliminated at the expense of a factor $1/(n/2)!$. The final result for this integral is obtained by expanding the determinant as a sum over permutations. By reordering the terms the result can be rewritten as

$$I_{10} = n! \mathrm{Pf} G \tag{25}$$

where the matrix elements of G are given by

$$\begin{aligned} G_{jk} &= -G_{kj} = \int dx dy w(x) w(y) (P_j(x) P_k(y) - P_k(x) P_j(y)) \\ &= \int dx dy w(x) w(y) \epsilon(x-y) P_j(x) P_k(y) \end{aligned} \tag{26}$$

and $\epsilon(x) = \frac{1}{2} x/|x|$.

In this case a natural scalar product is defined by

$$< f, g >_1 = \int dx dy w(x) w(y) \epsilon(x-y) f(x) g(y). \tag{27}$$

The corresponding orthogonal polynomials with orthogonality relations given by

$$< P_{2j}, P_{2k+1} >_1 = - < P_{2k+1}, P_{2j} >_1 = \delta_{jk}, \tag{28}$$

$$< P_{2j}, P_{2k} >_1 = < P_{2k+1}, P_{2j+1} >_1 = 0. \tag{29}$$

are called the skew orthogonal polynomials of the second kind.

3 Integration over part of the eigenvalues

In this section we consider the integrals

$$I_{\beta m} = \int |\Delta(\{x\})|^\beta \prod_{j=1}^{n} w(x_j) dx_{m+1} dx_n \tag{30}$$

for arbitrary value of m. Such integrals can be calculated recursively by using a theorem due to Dyson [6].

Theorem. Let $f(x,y)$ be a quaternion valued function satisfying the following conditions

- $f(x,y) = [f(y,x)]^D$,
- $\int dy f(x,y) f(y,z) = f(x,z) + \lambda f(x,z) - f(x,z)\lambda$,

where λ is an arbitrary constant quaternion. Then

$$\int dx_n w(x_n) \det |f(x_i, x_j)|_n = (c - n + 1) \det |f(x_i, x_j)|_{n-1}, \tag{31}$$

where

$$c = \int dx w(x) f(x, x), \tag{32}$$

and the determinant is a quaternion determinant as defined in (4).

The proof of this theorem follows by expansion of the quaternion determinant. The term c arises if x_n occurs in a one-cycle. However, $n-1$ of the terms where x_n occurs in a cycle of length two or longer result in the same determinant of order $n-1$ after using the second condition of the theorem. The terms proportional to λ cancel because of the linearity property of the determinant.

3.1 $\beta = 2$

Let us first consider the case $\beta = 2$. We rewrite the Vandermonde determinant using monic orthogonal polynomials satisfying the orthogonality relations

$$\int dx w(x) P_j(x) P_k(x) = h_j \delta_{jk}. \tag{33}$$

We then find

$$\begin{aligned} \Delta^2(\{x\}) &= \prod_{j=0}^{n-1} h_j \det |P_{j-1}(x_i)| \det |\frac{1}{h_{j-1}} P_{j-1}(x_i)| \\ &= \prod_{j=0}^{n-1} h_j \det |\sum_{l=0}^{n-1} \frac{1}{h_l} P_l(x_i) P_l(x_j)|, \end{aligned} \tag{34}$$

where the last expression has been obtained by multiplying the transpose of the first matrix with the second matrix. If we define

$$f(x, y) = \sum_{l=0}^{n-1} \frac{1}{h_l} P_l(x) P_l(x) \sqrt{w(x) w(y)}, \tag{35}$$

our integrals can be written as

$$I_{2m} = \int dx_{m+1} \cdots dx_n \det |f(x_i, x_j)|_n. \tag{36}$$

The function $f(x, y)$ satisfies the two conditions of Dyson's theorem. The first condition is satisfied trivially and the second condition follows immediately from the orthogonality relations for the orthogonal polynomials (in this case $\lambda = 0$). The constant $c = n$ which follows from the normalization of the orthogonal polynomials. Our final result for I_{2m} is thus given by

$$I_{2m} = (n - m)! \det |f_2(x_i, y_j)|_m. \tag{37}$$

3.2 $\beta = 4$

Next we consider the case $\beta = 4$. We introduce the quaternions χ_k with 2×2 matrix representation

$$C(\chi_k) = \begin{pmatrix} P_{2k}(x_k) & P_{2k+1}(x_k) \\ P'_{2k}(x_k) & P'_{2k+1}(x_k) \end{pmatrix}, \tag{38}$$

where $P_i(x)$ is a polynomial of degree i. The dual of this quaternion is given by

$$C(\chi_k^D) = \begin{pmatrix} P'_{2k+1}(x_k) & -P_{2k+1}(x_k) \\ -P'_{2k}(x_k) & P_{2k}(x_k) \end{pmatrix}. \tag{39}$$

The product of these two quaternions is given by

$$C(\chi_j^D, \chi_k) = \begin{pmatrix} < P_{2k}, P_{2j+1} > & < P_{2k+1}, P_{2j+1} > \\ < P_{2k}, P_{2j} > & < P_{2k+1}, P_{2j+1} > \end{pmatrix}, \tag{40}$$

where

$$< f, g >= fg' - gf'. \tag{41}$$

Next we choose as P_j the skew orthogonal polynomials of the first kind, i.e.

$$\int dx w(x) < P_{2j}, P_{2k+1} >= h_j \delta_{jk}, \tag{42}$$

$$\int dx w(x) < P_{2j}, P_{2k} >= 0, \tag{43}$$

$$\int dx w(x) < P_{2j+1}, P_{2k+1} >= 0. \tag{44}$$

For $\beta = 4$ the kernel function is defined by

$$f_4(x, y) = \sum_l \frac{1}{h_l} \chi_l(x) \chi_l^D(y) \sqrt{w(x)w(y)}. \tag{45}$$

This kernel function satisfies the conditions of Dyson's theorem. The duality relation is satisfied trivially, and the convolution property follows from the orthogonality relations of the skew orthogonal polynomials which in terms of the quaternions χ_k is given by

$$\int dx w(x) \chi_j^D \chi_k = \mathbf{1} h_j \delta_{jk}. \tag{46}$$

It turns out that also in this case $\lambda = 0$. The constant c also immediately follows from the orthogonality relation and is given by $c = n$. Recursively, we find the result for the integral for arbitrary values of m

$$I_{4m} = (n - m)! \det |f_4(x_i, y_j)|_m. \tag{47}$$

The calculation for $\beta = 1$ proceeds in a simular fashion. For details we refer to an extended version of these lectures [8].

4 Appendix

In this appendix we introduce quaternions and discuss some of their properties.

Quaternions are defined by

$$q = a_0 + a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3, \tag{48}$$

where the a_μ are real variables. If the a_μ are real the quaternion is called quaternion real. The quaternion units satisfy the relations

$$\epsilon_1^2 = \epsilon_2^2 = \epsilon_3^2 = -1, \tag{49}$$

$$\epsilon_1\epsilon_2 = -\epsilon_2\epsilon_1 = \epsilon_3, \tag{50}$$

and the cyclic permutations of the last relation. An explicit representation of the quaternion units is given by

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{51}$$

$$\mathbf{e_1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{52}$$

$$\mathbf{e_2} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{53}$$

$$\mathbf{e_3} = \begin{pmatrix} i & 0 \\ -i & 0 \end{pmatrix}. \tag{54}$$

The dual of a quaternion is defined by

$$q^D = a_0 - a_1\epsilon_1 - a_2\epsilon_2 - a_3\epsilon_3, \tag{55}$$

and the Hermitian conjugate is given by

$$q^\dagger = a_0^* - a_1^*\epsilon_1 - a_2^*\epsilon_2 - a_3^*\epsilon_3. \tag{56}$$

A quaternion matrix is selfdual if selfdual if $A_{ij} = A_{ji}^D$. It is Hermitian if $A_{ij} = A_{ji}^\dagger$. Therefore a Hermitian quarternion matrix that is self-dual is quaternion real.

A quaternion matrix A can also be represented in terms of on ordinary complex matrix with double size which we will denote by $C(A)$.

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