16 Symmetries of the QCD Partition Function

It is well-known that the QCD action is greatly constrained by gauge symmetry, Euclidean Poincaré invariance and renormalizability. These symmetries determine the structure of the Dirac operator. In this section we will discuss the global symmetries of the Dirac operator. They play an essential role in its spectral properties in the deepest infrared. In particular, the chiral symmetry, the flavor symmetry and the anti-unitary symmetry of the continuum Dirac operator are discussed.

16.1 The QCD partition function

The QCD partition function describing strong interactions in a box of volume $V_3 = L^3$ can be expressed in terms of the eigenvalues of the QCD Hamiltonian E_k as

$$Z^{QCD} = \sum_k e^{-\beta E_k}, \tag{1}$$

where β is the inverse temperature. At low temperatures ($\beta \to \infty$) the partition function is dominated by the lightest states of the theory, namely the vacuum state, with an energy density of E_0/V_3, and massless excitations thereof. The partition function Z_{QCD} can be rewritten as Euclidean functional integral over the nonabelian gauge fields A_μe

$$Z^{QCD} = \int dA_\mu \prod_{f=1}^{N_f} \det(D + m_f) e^{-S_{YM}}/g^2, \tag{2}$$

where S_{YM} is the Yang-Mills action given by

$$S_{YM} = \int d^4 d[-\frac{1}{4} F^a_{\mu\nu} 2 + i \frac{\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}]. \tag{3}$$

The field strength is given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{abc} A^b_\mu A^c_\nu. \tag{4}$$

The f_{abc} are the structure constants of the gauge group $SU(N_c)$. We also denote the gauge fields by

$$A_\mu = A_{\mu a} \frac{T^a}{2}, \tag{5}$$

where T^a are the generators of the gauge group. The dual of the field strength is given by

$$\tilde{F}_{\mu\nu} = \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}. \tag{6}$$

The integral $\nu \equiv \int d^4x \frac{1}{32\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}$ is a topological invariant, i.e. it does not change under continuous transformations of the gauge field. An important class of field configurations are instantons. These are topological nontrivial field configurations that minimize the classical action which are classified according to their topological charge ν.

The anti-Hermitian Dirac operator in (2) is given by

$$D = \gamma_\mu(\partial_\mu + iA_\mu), \tag{7}$$

where the γ_μ are the Euclidean Dirac matrices with anti-commutation relation

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \tag{8}$$

In the chiral representation the γ-matrices are given by

$$\gamma_k = \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix} \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

The integration measure can be defined by discretizing space time

$$dA_\mu^a = \prod_x dA_\mu^a(x). \tag{10}$$

A particular popular discretization is the lattice discritization where the QCD action is discretized on a hypercubic lattice with spacing a. Alternatively, we may define the measure in terms of a complete set of functions $\phi^a_{n\,\mu}$. In terms of the expansion

$$A_\mu^a = \sum_{n=1}^{N} a_n \phi^a_{n\,\mu} \tag{11}$$

the measure is defined as

$$\prod_n da_n. \tag{12}$$

A field theory is obtained by taking the continuum limit, i.e. the limit $a \to 0$ or $N \to \infty$ for the integration measures discussed above. This limit only exists is we simultaneously adjust the coupling constant, i.e. $g \to g(a)$ or $g(N)$. It turns out that this adjusted coupling constant apporaches zero in the continuum limit, a property known as asymptotic freedom. If such limit exists the field theory is called renormalizable.

16.2 Axial Symmetry

We will be mainly interested in the eigenvalues of the QCD Dirac operator. They are given by

$$D\phi_k = i\lambda_k \phi_k. \tag{13}$$

The eigenvalues are purely imaginary because the Dirac operator is anti-Hermitian (notice that $\partial_\mu^\dagger = -\partial_\mu$ and $A_\mu^\dagger = A_\mu$.

The chiral symmetry, or the $U_A(1)$ symmetry, can be expressed in terms of the anti-commutation relation

$$\{\gamma_5, D\} = 0. \tag{14}$$

This implies that all nonzero eigenvalues occur in pairs $\pm i\lambda_k$ with eigenfunctions given by ϕ_k and $\gamma_5\phi_k$. If $\lambda_k = 0$ the possibility exists that $\gamma_5\phi_k \sim \phi_k$, so that $\lambda_k = 0$ is an unpaired eigenvalue. According to the Atiyah-Singer theorem, the total number of such zero eigenvalues is a topological invariant, i.e., it does not change under continuous transformations of the gauge field configuration. Indeed, this possibility is realized by the field of an instanton which is a solution of the classical equations of motion. On the other hand, it cannot be excluded that $\lambda_k = 0$ while ϕ_k and $\gamma_5 \phi_k$ are linearly independent. However, this imposes additional constraints on the gauge fields which will be violated by infinitesimal deformations. Generically, such situation does not occur.

In a decomposition according to the total number of topological zero modes, the QCD partition function can be written as

$$Z^{QCD} = \sum_\nu e^{i\nu\theta} Z_\nu^{QCD}, \tag{15}$$

where

$$Z_\nu^{QCD} = \langle \prod_f m_f^\nu \prod_k (\lambda_k^2 + m_f^2) \rangle_\nu. \tag{16}$$

Here, $\langle \cdots \rangle_\nu$ denotes the average over gauge-field configurations with topological charge ν weighted by the Yang-Mills action. In we introduce right-handed and left-handed masses as complex conjugated masses we find that the θ dependence of the QCD partition function is only through the combination $me^{i\theta/N_f}$.

16.3 Flavor Symmetry

A second important global symmetry is the flavor symmetry. This symmetry can be best explained by writing the fermion determinant in the QCD partition function as a functional integral over Grassmann variables,

$$\prod_f \det(D + m_f) = \int d\psi d\bar\psi e^{\int d^4x \sum_{f=1}^{N_f} \bar\psi^f (D+m_f)\psi^f}. \tag{17}$$

In a chiral basis with $\psi_R = \gamma_5 \psi_R$ and $\psi_L = -\gamma_5 \psi_L$, this can be rewritten as

$$\prod_f \det(D + m_f) = \int d\psi d\bar\psi e^{\int d^4x [\sum_{f=1}^{N_f} \bar\psi_R^f D \psi_L^f + \bar\psi_L^f D \psi_R^f + \bar\psi_R^f m_f \psi_R^f + \bar\psi_L^f m_f \psi_L^f]}, \tag{18}$$

where we have used that

$$\bar\psi_R^f D \psi_R^f = \bar\psi_R^f \gamma_5 D \gamma_5 \psi_R^f = -\bar\psi_R^f D \psi_R^f = 0, \tag{19}$$

and similarly for the matrix elements between two left-handed states.

For $m_f = 0$ we have the symmetry

$$\begin{aligned} \psi_L &\to U\psi_L, \quad \bar\psi_R \to \bar\psi_R U^{-1}, \\ \bar\psi_L &\to \bar\psi_L V^{-1}, \quad \psi_R \to V\psi_R. \end{aligned} \tag{20}$$

The only condition to be imposed on U and V is that their inverse exists. If the number of lefthanded modes is equal to the number of right-handed modes we thus have an invariance under $Gl(N_f)_R \times Gl(N_f)_L$, where $Gl(N_f)$ is the group of complex $N_f \times N_f$ matrices with nonzero determinant. However, if the number of left-handed modes is not equal to the number of right-handed modes, the axial-symmetry group is broken to an $Sl(N_f)$ subgroup whereas the vector symmetry with $U = V$ remains unbroken. For $m = 0$ the flavor symmetry is thus broken explicitly to $Gl_V(N_f) \times Sl_A(N_f)$ by instantons or the anomaly.

What is much more important, though, is the spontaneous breaking of the axial flavor symmetry. From lattice QCD simulations and phenomenological arguments we know that the expectation value

$$\langle \bar{\psi}\psi \rangle = \langle \bar{\psi}_R \psi_R \rangle + \langle \bar{\psi}_L \psi_L \rangle \approx (240\, MeV)^3 \tag{21}$$

in the vacuum state of QCD instead of the symmetric possibility $\langle \bar{\psi}\psi \rangle = 0$. Phenomenologically, this is known because of the presence of Goldstone modes. The pions are much lighter than the σ mesons. The spontaneous breaking of the axial symmetry also follows from the absence of parity doublets. For example, the pion mass and the δ (or a_0) mass are very different ($m_\pi = 135 MeV$ and $m_\delta = 980 MeV$).

On easily verifies that $\langle \bar{\psi}\psi \rangle$ is only invariant for $U = V$. The vacuum state thus breaks the chiral symmetry down to $Gl_V(N_f)$. Notice that only the axial symmetries are broken. This is in agreement with the Vafa-Witten theorem [3] which states that vector symmetries cannot be broken spontaneously in vector-like theories such as QCD. We also observe that the *complete* axial group is broken. The reasons behind this maximum breaking [2] of chiral symmetry are less well understood.

16.4 Bosonic Quarks

Let us finally consider the case of bosonic quarks. For one bosonic flavor we have that

$$\frac{1}{\det(D+m)} = \int d\phi^* \phi e^{-\int d^4x \phi^*(D+m)\phi}, \tag{22}$$

where the ϕ are commuting fields. In a well-denfined theory the functional integral has to be convergent which is the case for positive m. The symmetries of the partition function should be compatible with these convergence requirement. In the presence case this implies that the symmetry transformations should not violate the complex conjugation structure of the integrtation variables. Let us study this in more detail. The $U(1)$ vector symmetry is given by

$$\begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} \to e^{i\theta} \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}, \tag{23}$$

$$\begin{pmatrix} \phi_R^* \\ \phi_L^* \end{pmatrix} \to e^{-i\theta} \begin{pmatrix} \phi_R^* \\ \phi_L^* \end{pmatrix}, \tag{24}$$

does not affect the complex conjugation structure. However, the axial $U(1)$ symmetry given by

$$\begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} \to \begin{pmatrix} e^{i\theta}\phi_R \\ e^{-i\theta}\phi_L \end{pmatrix}, \tag{25}$$

$$\begin{pmatrix} \phi_R^* \\ \phi_L^* \end{pmatrix} \to \begin{pmatrix} e^{i\theta}\phi_R^* \\ e^{-i\theta}\phi_L^* \end{pmatrix}, \tag{26}$$

violates the complex conjugation structure. After the transformation the integration variables ϕ and ϕ^* are no longer complex conjugate variables. We can repair this by considering imaginary angles, i.e. by considering the transformation

$$\begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} \to \begin{pmatrix} e^s \phi_R \\ e^{-s}\phi_L \end{pmatrix}, \tag{27}$$

$$\begin{pmatrix} \phi_R^* \\ \phi_L^* \end{pmatrix} \to \begin{pmatrix} e^s \phi_R^* \\ e^{-s}\phi_L^* \end{pmatrix} \tag{28}$$

The symmetry group is therefore not $U(1)$ but instead $Gl(1)/U(1)$.

For N_f flavors the vector symmetry is $U(N_f$. However, the axial symmetry is given by

$$\begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} \to \begin{pmatrix} e^H \phi_R \\ e^{-H}\phi_L \end{pmatrix}, \tag{29}$$

$$\begin{pmatrix} \phi_R^* \\ \phi_L^* \end{pmatrix} \to \begin{pmatrix} e^H \phi_R^* \\ e^{-H}\phi_L^* \end{pmatrix}, \tag{30}$$

where the matrix H contains only real elements. The symmetry group is thus given by the coset $Gl(N_f)/U(N_f$. An explicit parametrization of this coset is given by $AA^\dagger$ with $A \in Gl(N_f)$.

For bosonic quarks we also expect a maximum spontaneous breaking of the axial flavor symmetry, but with unbroken vector symmetries

For three or more colors the gauge fields are not related by complex conjugation and there are no anti-unitary symmetries. However, the situation is more interesting for $N_c = 2$ which will be discussed in the next chapter.

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