18 Symmetries QCD partition function for two colors

The QCD partition function for two colors is invariant under a larger group than the three-color QCD partition function. This well-known fact for QCD with fundamental fermions relies on the pseudo-real nature of $SU(2)$, i.e. the complex conjugate of an $SU(2)$ matrix parametrized by

$$U = a_0 + i a_k \sigma_k, \tag{1}$$

with the Pauli matrices denoted by σ_k and the a_μ real, is given by

$$U^* = \sigma_2 U \sigma_2. \tag{2}$$

18.1 Anti-Unitary Symmetry

Because of the pseudo-reality of $SU(2)$ we have that

$$A_\mu^* = -\tau_2 A_\mu \tau_2, \tag{3}$$

where the τ_k are the Pauli matrices acting in color space. Using the explicit representation for the γ-matrices one easily shows that

$$\gamma_\mu^* = -\gamma_2 \gamma_4 \gamma_m u \gamma_m u. \tag{4}$$

For the Dirac operator $iD = i\gamma_\mu \partial_\mu + \gamma_\mu A_\mu$ we thus have

$$\tau_2 \gamma_2 \gamma_4 (iD)^* \tau \gamma_2 \gamma_4 \tau_2 = iD. \tag{5}$$

This can also be written as

$$[KC\tau_2, iD] = 0, \tag{6}$$

where K is the complex conjugation operator, $C = \gamma_2\gamma_4$ is the charge conjugation matrix. Because

$$(KC\tau_2)^2 = 1 \tag{7}$$

we can always find a basis such that the Dirac matrix is real for any D. The proof is along the same lines as the proof that time reversal symmetry results in real matrix elements for the Hamiltonian in quantum mechanics.

18.2 Chiral Symmetries

The symmetries of the fermionic action can be best discussed by writing the determinant in the partition function as a Gaussian integral over the Grassmann fields ϕ and $\bar{\phi}$. Using the chiral representation of the γ matrices with

$$\gamma_\mu = \left(\begin{array}{cc} 0 & \hat{\sigma}_\mu \\ \hat{\sigma}_\mu^+ & 0 \end{array} \right), \tag{8}$$

and $\hat{\sigma}_\mu = (1, i\sigma_k)$ with σ_k the Pauli σ-matrices, the fermionic action can be written as

$$S_F = \int d^4x \sum_{f=1}^{N_f} \begin{pmatrix} \bar\phi_R \\ \bar\phi_L \end{pmatrix} \begin{pmatrix} m_f & \hat\sigma_\mu(\partial_\mu + iA_\mu) \\ \hat\sigma_\mu^+(\partial_\mu + iA_\mu) & m_f \end{pmatrix} \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}. \tag{9}$$

For $N_c = 2$, we have that $A_\mu^T = -\tau_2 A_\mu \tau_2$ (with $\tau_2 = \sigma_2$ in color space). Combining this with the relation $\hat\sigma_\mu^* = \sigma_2 \hat\sigma_\mu \sigma_2$ we find

$$\bar\phi_L^f \hat\sigma_\mu^+(\partial_\mu + iA_\mu)\phi_R^f = \phi_R^f \sigma_2\tau_2 \hat\sigma_\mu(\partial_\mu + iA_\mu)\sigma_2\tau_2 \bar\phi_L^f, \tag{10}$$

where we have used that ∂_μ is anti-Hermitean and that the fermion fields are anti-commuting Grassmann variables. The fermionic action can thus be rewritten as

$$S_F(m_f = 0) = \int d^4x \sum_{f=1}^{N_f} \begin{pmatrix} \bar\phi_R^f \\ \sigma_2\tau_2\phi_R^f \end{pmatrix} \begin{pmatrix} \hat\sigma_\mu(\partial_\mu + iA_\mu) & 0 \\ 0 & \hat\sigma_\mu(\partial_\mu + iA_\mu) \end{pmatrix} \begin{pmatrix} \phi_L^f \\ \sigma_2\tau_2\bar\phi_L^f \end{pmatrix}. \tag{11}$$

Obviously the symmetry group is $Gl(2N_f)$, i.e. the group of complex matrices with nonzero determinant. If the number of left-handed modes is not equal to the number of right-handed modes because of the anomaly, an axial $U(1)$ is broken explicitly as for three or more colors.

The mass term is given by

$$\begin{aligned} S_m &= \int d^4x \sum_{f=1}^{N_f} \left[\begin{pmatrix} \phi_L^f \\ \sigma_2\tau_2\bar\phi_L^f \end{pmatrix} \begin{pmatrix} 0 & -m_f\sigma_2\tau_2 \\ m_f\sigma_2\tau_2 & 0 \end{pmatrix} \begin{pmatrix} \phi_L^f \\ \sigma_2\tau_2\bar\phi_L^f \end{pmatrix} \right. \\ &+ \left. \begin{pmatrix} \bar\phi_R^f \\ \sigma_2\tau_2\phi_R^f \end{pmatrix} \begin{pmatrix} 0 & m_f\sigma_2\tau_2 \\ -m_f\sigma_2\tau_2 & 0 \end{pmatrix} \begin{pmatrix} \bar\phi_R^f \\ \sigma_2\tau_2\phi_R^f \end{pmatrix} \right]. \end{aligned} \tag{12}$$

A quark bilinear with nonzero expectation vacuum expectation value has to be scalar with respect to Lorentz transformations and gauge transformations. According to the Vafa-Witten theorem [1] vector symmetries are not spontenously broken so that a quark condensate has to invariant with respect to $U(N_f) \times U(N_f)$. Also in this case we expect maximum spontaneous chiral symmetry breaking consistent with the Vafa-Witten theorem [1]. This means that only the subgroup of $U(2N_f)$ that leaves both $\bar\phi_R\phi_R$ and $\bar\phi_L\phi_L$ invariant remains unbroken. Therefore, only the subgroup that leaves

$$\begin{pmatrix} \phi_L^f \\ \bar\sigma_2\tau_2\bar\phi_L^f \end{pmatrix} \begin{pmatrix} 0 & -\sigma_2\tau_2 \\ \sigma_2\tau_2 & 0 \end{pmatrix} \begin{pmatrix} \phi_L^f \\ \sigma_2\tau_2\bar\phi_L^f \end{pmatrix} \tag{13}$$

invariant remains unbroken ($\bar\phi_R\phi_R$ is invariant under the same transformations). This is the symplectic group $Sp(N_f)$. The Goldstone manifold is thus given by the coset $SU(2N_f)/Sp(N_f)$.

Physically, for $N_c = 2$ a baryon consists out of two quarks and is boson. Therefore the possibility exists that a baryon is a Goldstone boson. This is indeed the case. For example, for two flavors we have 15-10=5 Goldstone bosons. Three of them are the usual pions whereas the two remaining ones are a quark-quark state and its anti-particle.

18.3 Bosonic Quarks

As was the case for three colors for bosonic quarks we have to take into account the convergence of the integrals. We start from the fact that we can always find a representation in which the matrix elements of the Dirac operator are real. Let us denote the Dirac operator in this representation by C. An integral representation of the fermion determinant is given by

$$\frac{1}{\det\begin{pmatrix} im & C \\ C^T & m \end{pmatrix}} = \int dSdT \exp i \begin{pmatrix} S_R \\ S_L \\ T_R \\ T_L \end{pmatrix} \begin{pmatrix} im & C & 0 & 0 \\ C^T & im & 0 & 0 \\ 0 & 0 & im & C \\ 0 & 0 & C^T & im \end{pmatrix} \begin{pmatrix} S_R \\ T_R \\ S_L \\ T_L \end{pmatrix} \tag{14}$$

where all integration variables are real. In the functional representation we have exploited explicitly that the Dirac operator is a real symmetric matrix. The symmetry group that leaves the reality of the fields intact is $Gl(2N_f, R)$. It acts on the fields as

$$\begin{pmatrix} S_R \\ T_R \end{pmatrix} \to \begin{pmatrix} S_R \\ T_R \end{pmatrix} U^{-1} \qquad \begin{pmatrix} S_L \\ T_L \end{pmatrix} \to U \begin{pmatrix} S_L \\ T_L \end{pmatrix}. \tag{15}$$

In agreement with the Vafa-Witten theorem [1] we expect that the axial symmetries are spontanously broken, whereas the vector symmetries remain unbroken. The vector symmeries are the symmetries that leave the mass term invariant, i.e. the terms that leave

$$S_R S_R + T_R T_R + S_L S_L + T_L T_L \tag{16}$$

invariant. In terms of our symmetry transformation, this amounts to the condition that $U^T U = 1$, i.e. the symmetry is broken to $O(2N_f)$. The Goldstone manifold for N_f bosonic quarks is thus given by $Gl(2N_f)/O(2N_f)$.

References

[1] C. Vafa and E. Witten, Nucl. Phys. **B234** (1984) 173.

[2] J.J.M. Verbaarschot, Kyoto Lectures.