19 Infrared limit of the QCD Partition Function and Leutwyler-Smilga Sum-Rules

19.1 The chiral Lagrangian

For light quarks the low energy limit of QCD is well understood. It is given by the chiral Lagrangian describing the interactions of the pseudoscalar mesons. The reason is that pions are Goldstone bosons which are the only light degrees of freedom in a confining theory such as QCD. To lowest order in the quark masses and the momenta, the chiral Lagrangian is completely dictated by chiral symmetry and Lorentz invariance. In the case of N_f light quarks with chiral symmetry breaking according to $SU_L(N_f) \times SU_R(N_f) \to SU_V(N_f)$ the so called Weinberg Lagrangian is given by [7]

$$\mathcal{L}_{\rm eff}(U) = \frac{F^2}{4}\,{\rm Tr}(\partial_\mu U\,\partial_\mu U^\dagger) - \frac{\Sigma}{2}\,{\rm Tr}(\hat{\mathcal{M}}U + \hat{\mathcal{M}}U^\dagger), \tag{1}$$

where F is the pion decay constant, Σ is the chiral condensate and $\mathcal{M}$ is the quark mass matrix. The fields $U(x)$ are $SU(N_f)$ matrices parametrized as

$$U = \exp(i\sqrt{2}\Pi_a t^a/F), \tag{2}$$

with the generators of $SU(N_f)$ normalized according to ${\rm Tr} t^a t^b = \delta^{ab}$. The chiral Lagrangian has been used extensively for the description of pion-pion scattering amplitudes.

To lowest order in the pion fields we find for equal quark masses

$$\mathcal{L}_{\rm eff}(U) = \frac{1}{2}\partial_\mu \Pi^a \partial^\mu \Pi^a + \frac{\Sigma m}{F^2}\Pi^a\Pi^a. \tag{3}$$

This results in the pion propagator $1/(p^2+m_\pi^2)$ with pion mass given by the Gellmann-Oakes-Renner relation

$$m_\pi^2 = \frac{2m\Sigma}{F^2}. \tag{4}$$

In the long-wavelength limit the order of magnitude of the different terms contributing to the action of the chiral Lagrangian is given by [1]

$$S = \int d^4x \mathcal{L}(U) \sim L^{d-2}{\Pi^a_{NZM}}^2 + L^d\frac{\Sigma m}{F^2}({\Pi^a_{ZM}}^2 + {\Pi^a_{NZM}}^2). \tag{5}$$

Here, Π^a_{ZM} represents the zero momentum modes with no space time dependence, whereas the nonzero momentum modes are denoted by by $\Pi^a_{NZM}(x)$. This decomposition has two immediate consequences. First, for $\frac{\Sigma m}{F^2} \gg \frac{1}{V}$ the fluctuations of the pion fields are small and it is justified to expand U in powers of Π^a. Second, for

$$\frac{\Sigma m}{F^2} \ll \frac{1}{\sqrt{V}} \tag{6}$$

the fluctuations of the zero modes dominate the fluctuations of the nonzero modes, and only the contribution from the zero modes has to be taken into account for the calculation of an observable. In this limit the so called finite volume partition function is given by

$$Z^{\rm eff}_{N_f}(\mathcal{M},\theta) \sim \int_{U\in SU(N_f)} dU e^{V\Sigma {\rm Re\,Tr}\,\mathcal{M}Ue^{i\theta/N_f}}, \tag{7}$$

where the θ-dependence follows from the dependence of the QCD partition function on the combination [2] $me^{i\theta/N_f}$. We emphasize that any theory with the same pattern of chiral symmetry breaking as QCD can be reduced to the same extreme infrared limit.

The effective partition function at fixed ν follows by Fourier inversion

$$Z_\nu = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-i\nu\theta} Z(\theta). \tag{8}$$

Combining the integral over $SU(N_f)$ and the integral over $U(1)$ and using that

$$\det U e^{i\theta/N_f} = e^{i\theta}, \tag{9}$$

we find that

$$Z_\nu^{\rm eff} = \int_{U(N_f)} \det{}^\nu(U) e^{V\Sigma {\rm Re\,Tr}\,\mathcal{M} U e^{i\theta/N_f}}. \tag{10}$$

19.2 Leutwyler-Smilga Sum Rules

The Leutwyler-Smilga sum-rules [2] are obtained by expanding the partition function at fixed ν in powers of m before and after averaging over the gauge field configurations and equating the coefficients. This corresponds to an expansion in powers of m of both the QCD partition function and the finite volume partition function (10) in the sector of topological charge ν. As an example, we consider the coefficients of m^2 in the sector with $\nu = 0$. We expand the effective partition function to second order in the mass and use the group integrals

$$\int_{U\in U(n)} DU U_{ij}^p = 0, \tag{11}$$

$$\int_{U\in U(n)} DU U_{ij} U_{kl}^{-1} = \frac{1}{n}\delta_{il}\delta_{jk}. \tag{12}$$

$$\tag{13}$$

One easily verifies that the second integral is consistent with the unitarity relations for U. We then find

$$Z_0^{\rm eff}(m) = 1 + \frac{\Sigma^2 V^2}{4} m^2. \tag{14}$$

On the other hand, the expansion of the QCD partition function in powers of m is given by

$$\frac{Z_{QCD}^{\nu=0}(m)}{Z_{QCD}^{\nu=0}(0)} = 1 + m^2 \langle \sum_{\lambda_k>0} \frac{1}{\lambda_k^2} \rangle. \tag{15}$$

Equating the coefficients of m^2 results in the Leutwyler-Smilga sum-rule [2]

$$\langle {\sum}' \frac{1}{\lambda_k^2} \rangle = \frac{\Sigma^2 V^2}{4N_f}, \tag{16}$$

where the prime indicates that the sum is restricted to nonzero positive eigenvalues.

By equating higher powers of m^2 one can generate an infinite family of sum-rules for the eigenvalues of the Dirac operator. However, they are not sufficient to determine the Dirac spectrum. The reason is that the mass in the propagator also occurs in the fermion-determinant of the QCD partition function. However, as will be shown in the next chapter, the Dirac spectrum can be obtained from a chiral Lagrangian corresponding to QCD with additional bosonic and fermionic quarks [5]. We conclude that chiral symmetry breaking leads to correlations of the inverse eigenvalues which are determined by the underlying global symmetries.

19.3 The Banks-Casher relation and microscopic Spectral Density

In this section we give an interpretation of the Leutwyler-Smilga sum-rules in terms of the smallest eigenvalues of the QCD Dirac operator. We show that the smallest eigenvalues of the QCD Dirac operator are related to the chiral condensate by means of the Banks-Casher relation.

The order parameter of the chiral phase transition, $\langle\bar\psi\psi\rangle$, is nonzero only below the critical temperature or a critical chemical potential. As was shown by Banks and Casher [10], $\langle\bar\psi\psi\rangle$ is directly related to the eigenvalue density of the QCD Dirac operator per unit four-volume

$$\Sigma \equiv |\langle\bar\psi\psi\rangle| = \lim \frac{\pi\langle\rho(0)\rangle}{V}, \tag{17}$$

where the spectral density of the Dirac operator with eigenvalues $\{\lambda_k\}$ is defined by

$$\rho(\lambda) = \langle\sum_k \delta(\lambda-\lambda_k)\rangle. \tag{18}$$

It is elementary to derive this relation. The chiral condensate is defined as the logarithmic derivative of the QCD the partition function. For equal quark masses we obtain,

$$\langle\bar\psi\psi\rangle = -\lim \frac{1}{VN_f}\partial_m \log Z^{QCD}(m) = -\lim \frac{1}{V}\langle\sum_k \frac{2m}{\lambda_k^2+m^2}\rangle. \tag{19}$$

If we express the sum as an integral over the average spectral density, and take the thermodynamic limit before the chiral limit, so that many eigenvalues are less than m, we recover (17). The order of the limits in (17) is important. First we take the thermodynamic limit, next the chiral limit and, finally, the field theory limit. As follows from (19), the sign of $\langle\bar\psi\psi\rangle$ changes if m crosses the real axis.

An important consequence of the Bank-Casher formula (17) is that the eigenvalues near zero virtuality are spaced as

$$\Delta\lambda = 1/\rho(0) = \pi/\Sigma V. \tag{20}$$

This should be contrasted with the eigenvalue spectrum of the non-interacting Dirac operator. Then the eigenvalues are those of a free Dirac particle in a box with eigenvalue spacing equal to $\Delta\lambda \sim 1/V^{1/4}$ for the eigenvalues near $\lambda = 0$. Clearly, the presence of gauge fields leads to a strong modification of the spectrum near zero virtuality. Strong interactions result in the coupling of many degrees of freedom leading to extended states and correlated eigenvalues.

Let us study the Leutwyler-Smilga sum rule for equally spaced eigenvalues, i.e.

$$\lambda_n = \frac{\pi n}{\Sigma V}. \tag{21}$$

This results in the sum

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \frac{\Sigma^2 V^2}{6}. \tag{22}$$

We obtain the right parametric dependence of the Leutwyler-Smilga sum rules. The distribution of the smallest eigenvalues is sensitive to the number of massless quarks by means of the fermion determinant, and it is therefore not surprising that the exact result decreases with N_f.

Because the eigenvalues near zero are spaced as $\sim 1/\Sigma V$ it is natural to introduce the microscopic variable

$$u = \lambda V \Sigma, \tag{23}$$

and the microscopic spectral density [3]

$$\rho_s(u) = \lim_{V\to\infty} \frac{1}{V\Sigma} \langle \rho(\frac{u}{V\Sigma}) \rangle. \tag{24}$$

We expect that this limit exists and converges to a universal function which is determined by the global symmetries of the QCD Dirac operator. The calculation of this universal function from QCD is the main objective of these lectures. We will calculate $\rho_s(u)$ both from the simplest theory in this universality class which is chiral Random Matrix Theory (chRMT) and from the partial quenched chiral Lagrangian which describes the low-energy limit of the QCD partition function. We find that the two results coincide below the Thouless energy.

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