2 Random Matrix Theory

In this lecture we will discuss different types of random matrix ensembles. They were first introduced in physics by Wigner to describe the correlations of nuclear spectra. A rapid growth of the field occurred in the sixties mainly through work by Dyson and Mehta. Its growth was rekindled again in the late eighties and nineties both through the developments in mesoscopic physics and because of the applications of Random Matrix theory to quantum gravity. Both applications greatly enhanced the intellectual breath of the subject.

2.1 The Wigner-Dyson ensembles

The Wigner-Dyson ensembles are ensembles of hermitian matrices with matrix elements distributed according to

$$P(H)DH = \mathcal{N}e^{-\frac{N\beta}{4}\mathrm{Tr}H^\dagger H}DH. \tag{1}$$

Here, H is a Hermitian $N \times N$ matrix with *real*, *complex*, or *quaternion real* matrix elements, and β is the Dyson index of the ensemble. Its value is one for real matrix elements, it is two for complex matrix elements. The reason that it is four for quaternion real elements is that they can be expressed in terms of the Pauli spin matrices as

$$q = a_0 + ia_k\sigma_k, \tag{2}$$

where the a_μ are real variables. The measure DH is the Haar measure which is the product over independent differentials. The normalization constant of the probability distribution is denoted by N. The probability distribution is invariant under the transformations

$$H \to UHU^{-1}, \tag{3}$$

where U is an orthogonal transformation for $\beta = 1$, a unitary transformation for $\beta = 2$, and a symplectic transformations for $\beta = 4$. This is the reason why these ensembles are known as the *Gaussian Orthogonal Ensemble (GOE)*, the *Gaussian Unitary Ensemble (GUE)*, and the *Gaussian Symplectic Ensemble (GSE)*, respectively. Since both the eigenvalues of H and the Haar measure DH are invariant with respect to (3), the eigenvectors and the eigenvalues are independent with the distribution of the eigenvectors given by the the invariant measure of the corresponding orthogonal, unitary, or symplectic group. Notice that eigenvalues of a quaternion real matrix are quaternion scalars and therefore are doubly degenerate.

Exercise 2.1 Show that the symplectic ensemble is invariant under symplectic similarity transformations. A symplectic transformation is a unitary transformation that leaves the matrix Z invariant, where in quaternion notation, Z is $i\sigma_2$ times the identity.

Exercise 2.2 Show that the measure is invariant with respect to the similarity transformations (3). Give a separate proof for each value of β.

One way of deriving the probability distribution (1) is to look for a distribution that minimizes the information entropy under the constraints that the average, variance and normalization of the matrix elements is fix. The proof goes as follows. The sum of the information entropy and the constraints is given by

$$S = \int DH[-P(H)\log P(H) + aP(H) + bP(H)\mathrm{Tr}H + cP(H)\mathrm{Tr}H^2]. \tag{4}$$

Minimizing this with respect to $P(H)$ we obtain

$$-1 - \log P(H) + a + b\mathrm{Tr}H + c\mathrm{Tr}H^2 = 0, \tag{5}$$

resulting in a Gaussian distribution for $P(H)$.

The joint probability distribution of the Wigner-Dyson ensembles is given by

$$P(\{\lambda\})d\{\lambda\} = \mathcal{N}|\Delta(\{\lambda\})|^\beta \prod_k e^{-N\beta\lambda_k^2/4} d\lambda_k, \tag{6}$$

where the *Vandermonde determinant* is defined by

$$\Delta(\{\lambda\}) = \prod_{k>l}(\lambda_k - \lambda_l) \tag{7}$$

The Vandermonde determinant arises through the Jacobian from the transformation of the matrix elements as integration variables to the eigenvalues and eigenvector as integration variables. The distribution of the eigenvectors completely factorizes from the probability distribution. As an example, let us calculate the Jacobian for $\beta = 2$. One way to do this is to consider the invariant distance

$$\mathrm{Tr}dHdH^\dagger. \tag{8}$$

By differentiating (3) we find

$$U^{-1}dHU = \delta U\Lambda - \Lambda\delta U + d\Lambda, \tag{9}$$

where $\delta U = U^{-1}dU$. For the invariant distance we then obtain

$$\begin{aligned} \mathrm{Tr}dHdH^\dagger &= 2\mathrm{Tr}\delta U\Lambda\delta U\Lambda - 2\mathrm{Tr}\delta U\Lambda^2\delta U + \mathrm{Tr}d\Lambda^2 \\ &= 2\sum_{k<l}\delta U_{kl}\delta U^*_{kl}(\Lambda_k - \Lambda_l)^2 + \sum_k (d\Lambda_k)^2. \end{aligned} \tag{10}$$

We can immediately read of the Jacobian from the Lamé coefficients and the result is given by the square of the Vandermonde determinant.

Exercise 2.3 Calculate the Jacobian for $\beta = 1$ and $\beta = 4$.

2.2 The chiral ensembles

A second class of ensembles are the chiral ensembles with the chiral symmetry of the QCD Dirac operator. They are defined as the ensembles of $N \times N$ Hermitian matrices with block structure

$$H = \left(\begin{array}{cc} 0 & C \\ C^\dagger & 0 \end{array}\right) \tag{11}$$

and probability distribution given by

$$P(C)DC = \mathcal{N}\det{}^{N_f}\left(\begin{array}{cc} 0 & C \\ C^\dagger & 0 \end{array}\right) e^{-\frac{N\beta}{4}\mathrm{Tr}C^\dagger C} DC. \tag{12}$$

Again, DC is the Haar measure, and N_f is a real parameter (corresponding to the number of quark flavors in QCD). The matrix C is a rectangular $n \times (n+\nu)$ matrix. The nonzero eigenvalue of the matrix H occur in pairs $\pm\lambda_k$. If

$$\left(\begin{array}{cc} 0 & C \\ C^\dagger & 0 \end{array} \right) \left(\begin{array}{c} a \\ b \end{array} \right) = \lambda \left(\begin{array}{c} a \\ b \end{array} \right), \tag{13}$$

then

$$\left(\begin{array}{cc} 0 & C \\ C^\dagger & 0 \end{array} \right) \left(\begin{array}{c} a \\ -b \end{array} \right) = -\lambda \left(\begin{array}{c} a \\ -b \end{array} \right). \tag{14}$$

Generically, the matrix H in (11) has exactly $|\nu|$ zero eigenvalues. Also generically, the QCD Dirac operator corresponding to a field configuration with the topological charge ν has exactly $|\nu|$ zero eigenvalues, in accordance with the Atiyah-Singer index theorem. For this reason, ν is identified as the topological quantum number. The normalization constant of the probability distribution is denoted by $\mathcal{N}$. Also in this case one can distinguish ensembles with real, complex, or quaternion real matrix elements. They are denoted by $\beta = 1$, $\beta = 2$, and $\beta = 4$, respectively. The invariance property of the chiral ensembles is given by

$$C \to UCV^{-1}, \tag{15}$$

where U and V are orthogonal, unitary, and symplectic matrices, respectively. For this reason, the corresponding ensembles are known as the *chiral Gaussian Orthogonal Ensemble (chGOE)*, the *chiral Gaussian Unitary Ensemble (chGUE)*, and the *chiral Gaussian Symplectic Ensemble (chGSE)*, respectively.

Exercise 2.4 Show that the eigenvalues of the matrix with structure (11) occur in pairs $\pm\lambda$ or are zero. If C is a $n \times (n+\nu)$ block show that there are exactly ν zero eigenvalues.

It is always possible to decompose C as

$$C = U\Lambda V^{-1}, \tag{16}$$

where C is a diagonal matrix and U and V are orthogonal matrices for $\beta = 1$, unitary matrices for $\beta = 2$ and symplectic matrices for $\beta = 4$. The joint probability distribution for the eigenvalues is obtained by transforming to the eigenvalues and eigenvectors as new integration variables. The Jacobian can be obtained in the same way as in the previous section and is given by

$$J \sim \prod_k \lambda_k^{\beta\nu-1} \prod_{k<l} |(\lambda_k^2 - \lambda_l^2)|^\beta \tag{17}$$

resulting in a joint distribution of the eigenvalues

$$P(\{\lambda\})d\{\lambda\} = \mathcal{N}|\Delta(\{\lambda^2\})|^\beta \prod_k \lambda_k^\alpha e^{-N\beta\lambda_k^2/4} d\lambda_k, \tag{18}$$

where $\alpha = \beta - 1 + \beta\nu + 2N_f$. As for the Wigner-Dyson ensembles the distribution of the eigenvectors and the the distribution of the eigenvalues factorize.

Exercise 2.5 Find the Jacobian (17) using only symmetry arguments and dimensional arguments.

2.3 The superconducting ensembles

A third class of random matrix ensembles occurs in the description of disordered superconductors. Such ensembles with the symmetries of the Bogoliubov-de Gennes Hamiltonian have the block structure

$$H = \begin{pmatrix} A & B \\ B^\dagger & -A^T \end{pmatrix}, \tag{19}$$

where A is Hermitian and, depending on the underlying symmetries, the matrix B is symmetric or anti-symmetric. The probability distribution is given by

$$P(H)DH = \mathcal{N} \exp\left(-\frac{N\beta}{4}\mathrm{Tr}H^\dagger H\right) DH, \tag{20}$$

where DH is the Haar measure and $\mathcal{N}$ is a normalization constant. For symmetric B the matrix elements of H can be either complex ($\beta = 2$) or real ($\beta = 1$). One can easily very that in this case H satisfies the relation

$$H^T = -\Gamma_A H \Gamma_A, \tag{21}$$

where

$$\Gamma_A = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}. \tag{22}$$

One can then show that if ϕ is an eigenvector of H with eigenvalue λ, then $\Gamma_A\phi$ is an eigenvector of H^T with eigenvalue $-\lambda$. Because H and H^T have the same eigenvalues, all eigenvalues occur in pairs $\pm\lambda$.

For anti-symmetric B the matrix elements of H can be either complex ($\beta = 2$) or quaternion real ($\beta = 4$). In that case H satisfies the relation

$$H^T = -\Gamma_S H \Gamma_S, \tag{23}$$

where

$$\Gamma_S = \begin{pmatrix} 0 & 1_N \\ 1_N & 0 \end{pmatrix}. \tag{24}$$

From this relation it follows that if ϕ is an eigenvector of H with eigenvalue λ, then $\Gamma_S\phi$ is an eigenvector of H^T with eigenvalue $-\lambda$. Also in this case, all eigenvectors occur in pairs $\pm\lambda$.

The probability distribution of these so called Altland-Zirnbauer ensembles has the same general structure as the chiral ensembles. The value of $\alpha = 1$ for $\beta = 1$ and $\beta = 4$ whereas for $\beta = 2$ the value of α is 0 or 2.

Exercise 2.6 Derive the joint eigenvalue distribution for the Altland-Zirnbauer ensembles.

2.4 Anti-Unitary symmetries and the Dyson index

The value of the Dyson index is determined by the anti-unitary symmetries of the system. If there are no anti-unitary symmetries the Hamiltonian is Hermitean and the value of $\beta = 2$. In case there are anti-unitary symmetries given by

$$[KA, H] = 0, \tag{25}$$

with K a complex conjugation operator and A is an unitary matrix, we have to distinguish two cases:

$$(KA)^2 = 1 \quad \text{and} \quad (KA)^2 = -1, \tag{26}$$

corresponding to $\beta = 1$ and $\beta = 4$, respectively.

In the first case it is always possible to find a basis in which the Hamiltonian is real. Starting basis vector ϕ_1 we construct $\psi_1 = \phi_1 + KA\phi_1$. Then choose ϕ_2 perpendicular to ψ_1 and define $\psi_2 = \phi_1 + KA\phi_1$. Then

$$\begin{aligned} & (\phi_2 + KA\phi_2, \psi_1) \\ = \; & (KA\phi_2, \psi_1) \\ = \; & ((KA)^2\phi_2, KA\psi_1)^* \\ = \; & (\phi_2, \psi_1) = 0. \end{aligned} \tag{27}$$

The next basis vector is found by choosing ϕ_3 perpendicular to ψ_1 and ψ_2, etc.

In this base the Hamiltonian is real

$$\begin{aligned} H_{kl} = (\psi_k, H\psi_l) & \\ & = (KA\psi_k, KAH\psi_l)^* \\ & = (\psi_k, HKA\psi_l)^* \\ & = (\psi_k, H\psi_l)^* = H^*_{kl}. \end{aligned} \tag{28}$$

The best know anti-unitary operator in this class is the time-reversal operator. In that case A is the identity matrix.

In the case $(KA)^2 = -1$ all eigenvalues of the Hamiltonian are doubly degenerate. This can be show as follows. If ϕ_k is and eigenvector with eigenvalue λ_k, then it follows from the commutation relations that also $KA\phi_k$ is an eigenvector of the Hamiltonian with the same eigenvalue. The important thing is that this eigenvector is perpendicular to ϕ_k,

$$(\phi_k, KA\phi_k) = (KA\phi_k, (KA)^2\phi_k)^* = -(\phi_k, KA\phi_k). \tag{29}$$

One can prove that in this case it is possible to chose a basis in which the Hamiltonian matrix can be organized into real quaternions [11]. The eigenvalues of a Hermitean quaternion real matrix are quaternion scalars, and the eigenvalues of the original matrix are thus doubly degenerate in agreement with (29). The best known example of this Kramers degeneracy is for time reversal invariant systems of half-integral spin but no rotational invariance. Then the time reversal operator is given by $K\sigma_2$ with $(K\sigma_2)^2 = -1$.

Later in this lecture we will discuss other anti-unitary symmetries that enter in the Dirac operator for non-abelian Yang-Mills fields. They are relevant for the Dyson index of the chiral ensembles.

2.5 Classification of the Random Matrix Ensembles

The Random Matrix Ensembles in this section were classified according to the Cartan classification of symmetric spaces. A symmetric space is a manifold such that every point is a fixed point of an involutive isometry (i.e. $x_i \to -x_i$). The Riemann curvature tensor is covariantly constant in a symmetric space. A symmetric space is best characterized via the notion of symmetric pair. A symmetric pair (G, H) is defined as a pair of a connected Lie group G and a closed subgroup H such that an involutive analytic authomorphism σ of G exists with $H \in H_\sigma$, where H_σ is the set of fixed points of σ. Then, with some additional conditions (see the book by Helgason [10] for more details) the coset G/H is a symmetric space.

As an example, consider the group $U(p+q)$ and define Γ_{pq} as the diagonal matrix with the first p diagonal matrix elements equal to 1 and the remaining q diagonal matrix elements equal to -1. Consider the transformation

$$\sigma(g) = \Gamma_{pq} g \Gamma_{pq}, \tag{30}$$

which is an involution because obviously $\sigma(\sigma(g)) = g$. One can also easily show that it is an analytic automorphism. The set of fixed points of σ is the group $U(p) \times U(q)$. Therefore, $U(p+q)/U(p \times U(q)$ is a Riemannian symmetric space.

A symmetric space can be of the compact, noncompact or the Euclidean type with positive, negative or zero curvature, respectively. Each of the random matrix ensembles discussed in this section is tangent to one of the large classes of symmetric spaces. The complete classification is given in table I [9] where only the symmetric space of the compact type is given

Table 1: Random matrix ensemble, corresponding symmetric space, and the value for β.

RMT	symmetric space	compact Lie-group structure	β
GOE	AI	$U(N)/O(N)$	1
GUE	A	$U(N)$	2
GSE	AII	$U(2N)/Sp(N)$	4
chGOE	BDI	$SO(p+q)/SO(p) \times SO(q)$	1
chGUE	AIII	$U(p+q)/U(p) \times U(q)$	2
chGSE	CII	$Sp(p+q)/Sp(p) \times Sp(q)$	4
AZ-CI	CI	$Sp(N)/U(N)$	1
AZ-D	D	$SO(N)$	2
AZ-C	C	$Sp(N)$	2
AZ-DIII	DIII	$SO(2N)/U(N)$	4

For example, for special unitary matrices we can write $U = 1 + iH + \cdots$. Therefore, the Hermitean matrices are tangent to the space A (after dividing out a $U(1)$ factor). As another example, the generators of the class AIII are given by matrices with the structure (11).

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