20 Partial Quenching and the Valence Quark Mass Dependence of the Chiral Condensate

In this chapter we study the spectrum of the QCD Dirac operator by means of the valence quark mass dependence of the chiral condensate. For simplicity we will only consider the case with three or more colors and fundamental fermions.

20.1 The valence quark mass dependence of the chiral condensate

In terms of the eigenvalues of the Dirac operator the valence quark mass dependence of the chiral condensate is defined by [6, 7]

$$\begin{aligned}\Sigma(m_v; m_1, \cdots, m_{N_f}) &= \frac{1}{V}\sum_k \left\langle \frac{1}{i\lambda_k + m_v} \right\rangle \\ &= \frac{1}{V}\int d\lambda \frac{\rho(\lambda; m_1, \cdots, m_{N_f})}{i\lambda + m_v}. \end{aligned} \tag{1}$$

Here, $\langle \cdots \rangle$ denotes an average with respect to the distribution of the eigenvalues. The regular quark masses enter through the fermion determinant. The valence quark mass only enters in the resolvent and is therefore a direct probe of the Dirac spectrum.

The relation (1) can then be inverted to give $\rho(\lambda; m_1, \ldots, m_{N_f})$. As mentioned in [8], the spectral density follows from the discontinuity across the imaginary axis,

$$\mathrm{Disc}\big|_{m_v = i\lambda} \Sigma(m_v) = \lim_{\epsilon \to 0} \Sigma(i\lambda + \epsilon) - \Sigma(i\lambda - \epsilon) = 2\pi \sum_k \langle \delta(\lambda + \lambda_k) \rangle = 2\pi\rho(\lambda), \tag{2}$$

where we have suppressed the dependence on the sea-quark masses. This relation follows by writing $\Sigma(m_v)$ as a sum over eigenvalues.

20.2 Generating function

In this section we introduce a generating function for the valence quark mass dependence of the chiral condensate. This achieved by introducing the ratio of two determinants in the partition function. This corresponds to a Euclidean partition function of the form

$$Z^{\mathrm{pq}} = \int dA \, \frac{\det(D + m_{v1})}{\det(D + m_{v2})} \prod_{f=1}^{N_f} \det(D + m_f) \; e^{-S_{YM}} \, , \tag{3}$$

which we will call the partially quenched or pq-QCD partition function. When $m_{v1} = m_{v2}$ this partition function simply coincides with the original QCD partition function. However, it is now also the generator of a mass-dependent chiral condensate (see (1)) for the additional (say, fermionic) quark species. In the sector of topological charge ν we find

$$\Sigma(m_v; m_1, \cdots, m_{N_f}) = \frac{1}{V} \frac{\partial}{\partial m_{v1}}\bigg|_{m_{v1} = m_{v2} = m_v} \log Z_\nu^{\mathrm{pq}}. \tag{4}$$

Our aim is to find the chiral Lagrangian corresponding to (3). If we are successful, we have succeeded in deriving a partition function for the extreme infrared limit of the spectrum of the QCD Dirac operator. These questions will be addressed in the next sections.

20.3 The infrared limit of QCD

If we don't write the determinants in terms of integrals over complex conjugated variables, the global flavor symmetry of the partially quenched QCD partition function (3) is broken spontaneously according to

$$Gl_R(N_f+1|1) \times Gl_L(N_f+1|1) \to Gl_V(N_f+1|1) \tag{5}$$

with an axial $U(1)$ group broken explicitly by the anomaly. Here, the groups $Gl_R(N_f+1|1)$, $Gl_L(N_f+1|1)$ and $Gl_V(N_f+1|1)$ are super-groups of matrices acting on vectors with N_f+1 fermionic components and one bosonic component. The subscript refers to right-handed (R), left-handed (L) and vector (V), respectively. The latter transformations transform the right-handed and the left-handed fermion fields in the same way. For a confining theory such as QCD the only low-lying modes are the Goldstone modes associated with the spontaneous breaking of chiral symmetry. The quark masses play the role of symmetry breaking fields.

Although the axial supergroup $Gl_A(N_f+1|1)$ is a symmetry group of the pq-QCD action (3) it is not necessarily a symmetry of the QCD partition function. It may be that the symmetry transformations violate the convergence of the integrals in of the partition function. There are no problems for the Grassmann variables. However, the integrations over the bosonic fields are only convergent if the fields are related by complex conjugation. As we already had discussed for the case of bosonic quarks the $U_A(1)$ transformation in which the bosonic fields ϕ_R and ϕ_L are transformed according to a different phase factor is not a symmetry of the partition function. What is a symmetry of the partition function is the axial transformation

$$\phi_L \to e^s \phi_L, \qquad \phi_L^* \to e^s \phi_L^*, \tag{6}$$

$$\phi_R^* \to e^{-s} \phi_R^* \qquad \phi_R \to e^{-s} \phi_R. \tag{7}$$

Mathematically, this symmetry group is $Gl(1)/U(1)$. Had we restricted ourselves to the unitary subgroup $U(N_f+1|1)$ of $Gl(N_f+1|1)$ from the start, we would have missed this class of symmetry transformations.

Taking into account the chiral anomaly, the chiral symmetry in (3) is broken spontaneously according to (5). The symmetry of the QCD partition function is thus reduced to $Sl_V(N_f+1|1) \oslash Gl_V(1)$ where $\oslash$ denotes the semi-direct product. The Goldstone manifold corresponding to the symmetry breaking pattern (5) is based on the symmetric superspace $Sl_A(N_f+1|1)$. In our effective partition function the terms that break the axial symmetry will be included explicitly resulting in an integration manifold given by $Sl_A(N_f+1|1) \otimes Gl_A(1)$. However, this manifold is not a super-Riemannian manifold and is not suitable as an integration domain for the low energy partition function. As an integration domain we choose the maximum Riemannian submanifold of $Gl_A(N_f+1|1)$. This results in a fermion-fermion block given by the compact domain $U_A(N_f+1)$, whereas the boson-boson block is restricted to the non-compact domain $Gl(1)/U(1)$. Because of the super-trace, this compact/non-compact structure is required for obtaining a positive definite quadratic form for the mass term and the kinetic term of our low energy effective partition function. For a detailed mathematical discussion of this construction we refer to a paper by Zirnbauer [5].

To lowest order in the momenta, the infrared limit of the QCD partition function is uniquely determined by the geometry of the Goldstone manifold and Lorentz invariance. With the singlet field $\Phi_0 \equiv \mathrm{Str}\Phi$ fluctuating about the vacuum angle θ with an amplitude given by the singlet mass we thus obtain the effective partition function [12, 13]

$$\begin{aligned} Z(\theta, \hat{\mathcal{M}}) &= \int_{U \in Gl(N_f+1|1)} dU \exp \int d^4x \left[\frac{F^2}{4} \mathrm{Str}(\partial_\mu U \, \partial_\mu U^{-1}) \right. \\ &+ \left. \frac{\Sigma}{2} \mathrm{Str}(\hat{\mathcal{M}} U + \hat{\mathcal{M}} U^{-1}) + \frac{F^2 m_0^2}{12} (\frac{\sqrt{2}\Phi_0}{F} - \theta)^2 \right]. \end{aligned} \tag{8}$$

This partition function contains terms up to order p^2. The kinetic term of the singlet field is subleading and has not been included.

20.4 Domains in (Partially-Quenched) Chiral Perturbation Theory

In chiral perturbation theory the different domains of validity where analyzed in detail by Gasser and Leutwyler [3]. A similar analysis applies to partially quenched chiral perturbation theory. The idea is as follows. The U field can be decomposed as [3]

$$U = U_0 e^{i\psi(x)}. \tag{9}$$

where U_0 is a constant (zero-momentum) field. The kinetic term for the ψ fields is given by

$$\frac{1}{2} \int d^4x \partial_\mu \psi^a \partial_\mu \psi^a \sim L^2 \psi^2. \tag{10}$$

We observe that the magnitude of the fluctuations of the ψ field are of order $1/L$ which justifies a perturbative expansion of $\exp(i\psi(x))$. The fluctuations of the zero modes, on the other hand, are only limited by the mass term

$$\frac{1}{2} V \Sigma \mathrm{Str} \mathcal{M} (U_0 + U_0^{-1}). \tag{11}$$

For quark masses $m \gg 1/V\Sigma$, the field U_0 fluctuates close to the identity and the U_0 field can be expanded around the identity as well. This is the domain of chiral perturbation theory.

For valence quark masses in the range

$$\frac{1}{V\Sigma} \ll m_v \ll \frac{F^2}{\Sigma L^2} \tag{12}$$

the valence quark mass dependence of the chiral condensate is dominated by the zero momentum modes which can be treated perturbatively. Below we will show that in this domain chiral perturbation theory and random matrix theory coincide.

In the Dirac spectrum we can distinguish three important scales. First, the scale of the smallest eigenvalue,

$$\lambda_{rmmin} = \frac{\pi}{\Sigma V}. \tag{13}$$

Second, the valence quark mass corresponding to a valence quark Goldstone boson with Compton wavelength equal to the size of the box

$$m_c = \frac{F^2}{\Sigma L^2}. \tag{14}$$

Third, the scale of Λ_{QCD} which sets the mass scale of QCD. Based on these scales we can distinguish four different domains. In the domain where m_v is of the order of $\lambda_{\rm min}$ or less we have to take into account quantum fluctuations to all orders. For $\lambda_{\rm min} \ll m_v \ll m_c$, the quantum fluctuations of the field can be treated semi-classically, or in field theory language, by a loop expansion. Since the time scale conjugate to m_c is larger than the diffusion time across the length of the box, this domain is known as the ergodic domain. For $m_c \ll m_v \ll \Lambda_{QCD}$, chiral perturbation theory still applies, but the zero momentum modes no longer dominate the partition function. In this domain, the Dirac spectrum differs from chRMT. For $m_v \gg \Lambda_{QCD}$, the masses of the Goldstone modes and the other hadronic excitations are of the same order of magnitude. Chiral perturbation theory is no longer applicable, and one has to take into account the full QCD partition function.

In the theory of disordered mesoscopic systems the scale below which random matrix theory is valid is known as the Thouless energy and is given by

$$E_c = \frac{\hbar D}{L^2}, \tag{15}$$

where D is the diffusion constant for the diffusive motion of electrons in a disordered sample. The time conjugate to E_c is the time scale over which an electron diffuses across the sample. The time scale in mesoscopic physics corresponding to Λ_{QCD} is the elastic scattering time. The domain in-between E_c and $\hbar/\tau_e$ is known as the diffusive domain. This domain is characterized by diffusive motion of electrons in the disordered sample.

20.5 Zero momentum limit of the effective partition function

For valence quark masses in the domain (12) the valence quark mass dependence of the partition function of the QCD partition function is given by

$$Z_{\rm eff}(\theta, \hat{\mathcal{M}}) = \int_{U \in Gl(N_f+1|1)} dU e^{V \frac{\Sigma}{2} {\rm Str}(\hat{\mathcal{M}} U + \hat{\mathcal{M}} U^{-1}) - \frac{F^2 m_0^2 V}{12} (\frac{\sqrt{2}\Phi_0}{F} - \theta)^2}. \tag{16}$$

The partition function in a sector of topological charge ν follows by Fourier inversion [4]

$$Z^\nu_{\rm eff}(\hat{\mathcal{M}}) \quad = \quad \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\nu\theta} Z_{\rm eff}(\theta, \hat{\mathcal{M}}). \tag{17}$$

We perform the integration over θ after a Hubbard-Stratonovitch transformation which linearizes the singlet mass term. Up to a mass independent factor this results in the partition function [8]

$$Z^\nu_{\rm eff}(\hat{\mathcal{M}}) = \int_{U \in Gl(N_f+1|1)} dU \, {\rm Sdet}^\nu U \, e^{V \frac{\Sigma}{2} {\rm Str}(\hat{\mathcal{M}} U + \hat{\mathcal{M}} U^{-1})}. \tag{18}$$

In the next section we will study this partition function in the quenched limit ($N_f = 0$) and show that the valence quark mass dependence of the chiral condensate coincides with the result obtained from chiral Random Matrix Theory.

20.6 Nonperturbative evaluation of $\Sigma(m_v)$ in the quenched limit

In this section we calculate the valence quark mass dependence of the chiral condensate for the simplest case of $N_f = 0$ and $\nu = 0$ in the domain $m_v \ll F^2/\Sigma L^2$. In this domain the partition function is given by

$$Z(J) = \int_{U \in Gl(1|1)} dU \exp\left[\frac{\Sigma V}{2} \mathrm{Str}\begin{pmatrix} m_v + J & 0 \\ 0 & m_v \end{pmatrix}(U + U^{-1})\right], \tag{19}$$

where the integration is over the maximum super-Riemannian sub-manifold of $Gl(1|1)$. This manifold is parametrized by

$$U = \exp\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{s} \end{pmatrix} \tag{20}$$

The integration measure is the Haar measure which in terms of this parameterization it is given by

$$\mathrm{Sdet}\frac{\delta U_{kl}}{\delta\phi\, \delta s\, \delta\alpha\, \delta\beta}\; d\alpha d\beta d\phi ds, \tag{21}$$

where $\delta U \equiv U^{-1} dU$.

It is straightforward to calculate the Berezinian going from the variables $\delta U_{11}\, \delta U_{22}\, \delta U_{12}\, \delta U_{21}$ to the variables $\delta\phi\, \delta s\, \delta\alpha\, \delta\beta$. The derivative matrix is given by

$$B = \frac{\delta U_{kl}}{\delta\phi\, \delta s\, \delta\alpha\, \delta\beta} = \begin{pmatrix} i & 0 & \frac{\beta}{2} & \frac{\alpha}{2} \\ 0 & 1 & \frac{\beta}{2} & \frac{\alpha}{2} \\ 0 & 0 & e^{s-i\phi} & 0 \\ 0 & 0 & 0 & e^{-s+i\phi} \end{pmatrix}. \tag{22}$$

Using the definition of the graded determinant one simply finds that $\mathrm{Sdet}B = i$. Up to a constant, the integration measure is thus given by $d\phi ds d\alpha d\beta$. In general, for $N_f \neq 0$, the Berizinian is more complicated [15, 9].

We also need

$$\frac{1}{2}(U + U^{-1}) = \begin{pmatrix} \cos\phi(1 + \frac{\alpha\beta}{2}) & \alpha(e^{s} - e^{-i\phi}) \\ \beta(e^{i\phi} - e^{-s}) & \cosh s(1 - \frac{\alpha\beta}{2}) \end{pmatrix}. \tag{23}$$

After differentiating with respect to the source term ($\Sigma(m_v) = \partial_J \log Z(J)|_{J=0}/V$)this results in

$$\frac{\Sigma(m_v)}{\Sigma} = \int \frac{d\phi ds d\alpha d\beta}{2\pi} \cos\phi(1 + \frac{\alpha\beta}{2}) e^{x\cos\phi(1+\frac{\alpha\beta}{2}) - x\cosh s(1-\frac{\alpha\beta}{2})}. \tag{24}$$

With the Grassmann integral given by the coefficient of $\alpha\beta$ we obtain

$$\begin{aligned} \frac{\Sigma(m_v)}{\Sigma} &= \int \frac{ds d\phi}{4\pi}[\cos\phi \\ &+ x(\cos\phi + \cosh s)\cos\phi] e^{x(\cos\phi - \cosh s)}. \end{aligned} \tag{25}$$

Now all integrals can be expressed in terms of modified Bessel functions. We find

$$\frac{\Sigma(m_v)}{\Sigma} = I_1(x)K_0(x) + \frac{x}{2}(I_2(x)K_0(x) + I_0(x)K_0(x) + 2I_1(x)K_1(x)). \tag{26}$$

After using the recursion relation for modified Bessel functions,

$$I_2(x) = I_0(x) - \frac{2}{x} I_1(x), \tag{27}$$

we arrive at the final result [7, 8, 9]

$$\frac{\Sigma(m_v)}{\Sigma} = x(I_0(x)K_0(x) + I_1(x)K_1(x)) \tag{28}$$

where $x = mV\Sigma$.

This calculation can be generalized to arbitrary N_f and arbitrary ν. The calculation for arbitrary N_f is much more complicated, but with a natural generalization of the factorized parameterization, and using some known integrals over the unitary group, one arrives at an expression in terms of modified Bessel functions

$$\frac{\Sigma(x)}{\Sigma} = x(I_a(x)K_a(x) + I_{a+1}(x)K_{a-1}(x)), \tag{29}$$

where $a = N_f + |\nu|$. This result is in complete agreement [9] with chRMT to be discussed below.

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