3 The Orthogonal Polynomial Method

The main task of Random Matrix Theory is to derive the eigenvalue density and the spectral correlations of the eigenvalues from the joint eigenvalue distribution discussed in previous lecture. One of the methods to perform the integrals over the eigenvalues is the orthogonal polynomial method. It is particular useful for $\beta = 2$. The central idea is to express the Vandermonde determinant in terms of orthogonal polynomials and perform the integrations by means of orthogonality relations.

3.1 The Orthogonal Polynomials Method

Let us rescale the eigenvalues such that the joint probability distribution is given by

$$\rho(x_1, \cdots, x_N) = |\Delta(\{x_k\})|^2 e^{-N\sum_{k=1}^N V(x_k)} \tag{1}$$

The simplest cases is the GUE with $V(x) = x^2$, and the chGUE with $V(x) = x^2 - a\log x$. However, since a general potential does not complicate the expressions, we will derive our results for arbitrary $V(x)$. The Vandermonde determinant can be written as

$$\Delta(\{x_k\}) = \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_N \\ \vdots & & \vdots \\ x_1^{N-1} & \cdots & x_N^{N-1} \end{vmatrix} \tag{2}$$

By the addition of multiples of rows of lower order which do not change the determinant, each of the rows can be expressed in terms of orthogonal polynomials with respect to the weight function $\exp(-V(x))$, i.e.

$$\Delta(\{x_k\}) = \mathcal{N} \begin{vmatrix} p_0(x_1) & \cdots & p_0(x_N) \\ p_1(x_1) & \cdots & p_1(x_N) \\ \vdots & & \vdots \\ p_{N-1}(x_1) & \cdots & p_{N-1}(x_N) \end{vmatrix}. \tag{3}$$

The orthogonal polynomials are defined by

$$\int_{-\infty}^{\infty} dx e^{-NV(x)} p_k(x) p_l(x) = \delta_{kl}. \tag{4}$$

For the GUE the corresponding orthogonal polynomials are the Hermite polynomials. The normalization constant arises because in general the polynomials are not monic. It can be determined from the overall normalization of the distribution function.

It is convenient to introduce wave-functions defined by

$$\varphi_k(x) = p_k(x) e^{-NV(x)/2}. \tag{5}$$

The joint probability distribution can then be written as

$$\rho(x_1, \cdots, x_N) = \mathcal{N} \begin{vmatrix} \varphi_0(x_1) & \cdots & \varphi_{N-1}(x_1) \\ \vdots & & \vdots \\ \varphi_0(x_N) & \cdots & \varphi_{N-1}(x_N) \end{vmatrix} \times \begin{vmatrix} \varphi_0(x_1) & \cdots & \varphi_0(x_N) \\ \vdots & & \vdots \\ \varphi_{N-1}(x_1) & \cdots & \varphi_{N-1}(x_N) \end{vmatrix}. \tag{6}$$

By multiplying the two determinants, we find that joint eigenvalue distribution can be expressed as

$$\rho(x_1, \cdots, x_N) = \mathcal{N} \det_{ij}[K(x_i, x_j)] \tag{7}$$

where the kernel is defined by

$$K(x_i, x_j) = \sum_{k=0}^{N-1} \varphi_k(x_i)\varphi_k(x_j). \tag{8}$$

The kernel has the following elementary properties

$$\int K(x, x) = N, \qquad \int dy K(x, y)K(y, z) = K(x, z). \tag{9}$$

The spectral density is obtained by integrating over all eigenvalues except one. Expressing the determinant as a sum over all permutation of products of N matrix elements, one observes that in each term each argument occurs twice. Performing the integrations by means of (9) we arrive at the following correctly normalized expression for the spectral density

$$\rho(x) = K(x, x). \tag{10}$$

The two-point correlation function is obtained by integrating over all eigenvalues except two. In this case one finds

$$R_2(x, y) = -K(x, y)^2 + K(x, x)K(y, y). \tag{11}$$

Here, we have used that the total number of permutations for which 1 and 2 are in the same cycle is the same of the total number of permutations with 1 and 2 in different cycles. This can be seen as follows. In a cycle decomposition, a permutation with 1 and 2 in the same cycle can be written as $(1p2q)\pi_r$ or $(1q2p)\pi_r$ for a sequence of numbers p and q and remaining cycles π_r. For 1 and 2 in different cycles we have the permutations $(1p)(2q)\pi_r$ and $(1q)(2p)\pi_r$. From the length of the cycles one immediately finds that the signature of the two groups of permutations is opposite. It is also easy to see that the numerical factor for given p, q and π_r is the same is the two groups. This result determines (11) up to a overall constant. We observe that the correlation function vanishes for $x = y$ which also follows from the determinantal structure of the correlation function.

To find the proportionality constant we notice that

$$\int dx R_1(x) dx = N, \tag{12}$$

$$\int dx dy R_2(x, y) = N(N-1), \tag{13}$$

where the factor $(N-1)$ arise because all eigenvalues in the joint probability distribution are necessarily different. The proportionality constant is thus as given in (11).

3.2 The orthogonal polynomial method for the GUE

In case of the GUE the potential is given by $V(x) = x^2$. The corresponding orthogonal polynomials are the Hermite polynomials. The correctly normalized wave functions are given by

$$\frac{N^{1/4}}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-N\frac{x^2}{2}}. \tag{14}$$

The spectral density is thus given by

$$\rho(x) = \sum_{n=0}^{N-1} \frac{\sqrt{N}}{2^n \sqrt{\pi} n!} H_n(x\sqrt{N}) H_n(x\sqrt{N}) e^{-Nx^2}. \tag{15}$$

This sum can be calculated exactly by means of the Christoffel-Darboux formula

$$\sum_{n=0}^{N-1} \frac{H_n(y)H_n(y)}{2^n n!} = \frac{1}{2^N (N-1)!}[H_N(y)H_N(y) - H_{N-1}(y)H_{N+1}(y)]. \tag{16}$$

In the limit $N \to \infty$ we can use the asymptotic limit of the Hermite polynomials given by [**?**]

$$\frac{N^{1/4} H_{N-k}(x\sqrt{N}) e^{-N\frac{x^2}{2}}}{(2^{N-k}(N-k)!\sqrt{\pi})^{1/2}} = \sqrt{\frac{\sqrt{2}}{\pi \sin\phi}} \cos(N(\phi - \sin(2\phi)/2) - (k - \frac{1}{2})\phi - \frac{\pi}{4}) + O(1/N). \tag{17}$$

Here, $\cos\phi = x/\sqrt{2}$. This asymptotic result is valid for $N \to infty$ and k finite. If we use that

$$\cos\alpha \cos\alpha - \cos(\alpha+\beta)\cos(\alpha-\beta) = \sin^2\beta, \tag{18}$$

we find for the spectral density to $O(1/N)$

$$\rho(x) = \frac{N\sqrt{2}}{\pi \sin\phi} \sin^2\phi = \frac{N}{\pi}(2 - x^2)^{1/2}. \tag{19}$$

We observe that the average spectral density has the shape of a semicircle, and is correctly normalized to N.

Next we evaluate the two-point correlation function. In this case we use the Christoffel-Darboux formula for different values of the argument

$$\sum_{n=0}^{N-1} \frac{H_n(x)H_n(y)}{2^n n!} = \frac{1}{2^N (N-1)!} \frac{H_N(x)H_{N-1}(y) - H_{N-1}(y)H_N(x)}{x - y}. \tag{20}$$

We study the spectral correlation function in the microscopic limit, i.e. in the limit $N \to \infty$ with $(x-y)N$ fixed. Keeping only terms to leading order in $1/N$ we find for the kernel

$$\begin{aligned} K(x,y) = & \frac{1}{\pi \sin\phi}[\cos(N\zeta(\phi) + \phi/2 - \frac{\pi}{4})\cos(N\zeta(\phi') - \phi/2 - \frac{\pi}{4}) \\ - & \cos(N\zeta(\phi') + \phi/2 - \frac{\pi}{4})\cos(N\zeta(\phi) - \phi/2 - \frac{\pi}{4})], \end{aligned} \tag{21}$$

where

$$\zeta(\phi) = \phi - \sin(2\phi)/2, \tag{22}$$

$$\cos\phi = x/\sqrt{2}, \tag{23}$$

$$\cos\phi' = y/\sqrt{2}. \tag{24}$$

Using an addition formula for the goniometric functions we find

$$K(x,y) = \frac{1}{\pi\sin\phi}\frac{\sin(N\zeta(\phi') - N\zeta(\phi))\sin\phi}{x-y}. \tag{25}$$

By inspection we find that the function $\zeta(\phi)$ satisfies

$$\frac{d\zeta}{dx} = -\pi\rho(x)/N, \tag{26}$$

where

$$\rho(x) = \frac{N}{\pi}\sqrt{2-x^2}. \tag{27}$$

In the microscopic limit x and y are close and we can make the approximation

$$N\zeta(\phi') = N\zeta(\phi) + N(y-x)\frac{d\zeta}{dx}. \tag{28}$$

For the kernel we thus find

$$K(x,y) = \frac{1}{\pi}\frac{\sin\pi(x-y)\rho(x)}{x-y} \tag{29}$$

The unfolded distance between two eigenvalues is given by

$$r = (x-y)\rho(x). \tag{30}$$

This results in the unfolded two-point correlation function

$$Y_2(r) = \frac{sin^2\pi r}{\pi^2 r^2}. \tag{31}$$

The number variance is then given by

$$\Sigma^2(n) = n - 2\int_0^n (n-r)Y_2(r). \tag{32}$$

The first terms of the r.h.s. cancels because $\int_0^\infty dr Y_2(r) = 1/2$. The remaining integral is given by

$$\Sigma_2(n) = \frac{1}{\pi^2}(\log(2\pi n) + \gamma + 1), \tag{33}$$

where the logarithmic dependence follows from the asymptotic behavior of $Y_2(r)$ (with $\sin^2(\pi r)$ replaced by its average).

Excercise 3.2. Show this result from the expression for the number variance given in lecture 1.

3.3 Orthogonal polynomial method for the chGUE

In this case the potential is given by

$$V(x) = x^2 - a \log x. \tag{34}$$

For convenience we don't a factor N in the probability distribution. Since the eigenvalues are positive we can introduce new variable by $y = x^2$. In terms of these variables the orthogonal polynomials are given by

$$\int_0^\infty dx x^\alpha e^{-x} P_n^\alpha(x) P_m^\alpha(x) = h_n^\alpha \delta_{mn} \tag{35}$$

where we have introduced by $\alpha = (a-1)/2$ and the normalization is given by

$$h_n^\alpha = \left(\begin{array}{c} n+\alpha \\ n \end{array} \right) \tag{36}$$

These polynomials are well-known. They are the Laguerre polynomials, i.e. $P_n^\alpha(x) = L_n^\alpha(x)$. The spectral density is thus given by

$$\rho(x) = \sum_{n=0}^{N-1} \frac{L_n^\alpha(x) L_n^\alpha(x)}{h_n^\alpha} x^\alpha e^{-x}. \tag{37}$$

In this case the eigenvalues occur in pairs $\pm\lambda$. It thus makes sense to study the spectrum on the scale of the individual level spacings near $\lambda = 0$. We then can introduce the microscopic variable

$$u = \pi \lambda \rho(0), \tag{38}$$

and the microscopic spectral density defined by

$$\rho_s(u) = \frac{1}{\pi\rho(0)} \rho(\frac{u}{\pi\rho(0)}). \tag{39}$$

In the next lecture we will show that this limiting function is universal and therefore will be found in many systems with the symmetries of the chiral ensembles.

In terms of the orthogonal polynomials, the existence of the microscopic limit is based on the following scaling limit for the Laguerre polynomials.

$$\lim_{n\to\infty} \frac{1}{n^\alpha} L_n^\alpha(\frac{x}{n}) = x^{-\frac{\alpha}{2}} J_\alpha(2\sqrt{x}). \tag{40}$$

We will study the spectral density in the limit $N \to \infty$ with xN fixed. Since most terms contributing to the sum in the spectral density are for very large n, we can use the the asymptotic limit of the Laguerre polynomials. We find

$$\rho(x) = \sum_{n=0}^{N-1} J_\alpha(2\sqrt{xn}), \tag{41}$$

where $\exp(-x) \approx 1$ in the limit under consideration. The sum over n can be performed by replacing it by an integral over the integration variable $t = n/N$

We use the following integral for Bessel functions

$$\int_0^z dx x J_\nu^2(rx) = \frac{z^2}{2}(J_\nu^2(rz) - J_{\nu+1}(rz)J_{\nu-1}(rz)). \tag{42}$$

This results in

$$\rho(x) = N(J_\nu^2(2\sqrt{Nx}) - J_{\nu+1}(2\sqrt{Nx})J_{\nu-1}(2\sqrt{Nx})). \tag{43}$$

In terms of our original variable $y = \sqrt{x}$ we find

$$\rho(y) = 2Ny(J_\nu^2(2y\sqrt{N}) - J_{\nu+1}(2y\sqrt{N})J_{\nu-1}(2y\sqrt{N})). \tag{44}$$

Asymptotically, for $y \gg 1/\sqrt{N}$ but $y \ll 1$ we can use the asymptotic result

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\cos(z - \frac{\pi}{2}\nu - \frac{\pi}{4}). \tag{45}$$

resulting in the smoothened spectral density near zero

$$\rho(0) \sim \frac{2\sqrt{N}}{\pi}. \tag{46}$$

The microscopic variable is thus given by $u = 2y\sqrt{N}$ and the microscopic spectral density is given by

$$\rho_s(u) = \frac{u}{2}(J_\nu^2(u) - J_{\nu+1}(u)J_{\nu-1})). \tag{47}$$

In the next lecture we will show that this is a truly universal result. Later in these lecture we will discuss several physical examples for which this microscopic spectral density has been found.

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