4 Universality of the microscopic spectral density

In this lecture we show that the microscopic spectral density does not depend on the probability distribution of the the random matrices. In this chapter universality will be studied for the case of an invariant potential with a probability distribution that can be expressed in terms of the eigenvalues of the random matrix. However, universality has also been shown for deformations of the probability distribution that violate the unitary invariance of the RMT but that case is mathematically much more complicated. Later in these lectures we will return to such examples.

4.1 Universality of the microscopic spectral density for the chGUE

Below I will discuss universality using Shohat's method following a proof given by Akemann et al. [1]. The idea is to derive a differential relation from the recursion relation for the orthogonal polynomials in the microscopic limit.

We will find that this differential equation does not depend on the potential provided that microscopic variables are used. We will consider the chGUE with $\alpha = 0$. Then the orthogonal polynomials are defined by

$$\int_0^\infty dx P_k(x) P_l(x) e^{-nV(x)} = h_k \delta_k l, \tag{1}$$

where x denotes the square of the eigenvalues. We normalize the orthogonal polynomials according to $P_k(0) = 1$, the same as in the case of the Laguerre polynomials with $V(x) = x$. All orthogonal polynomials satisfy a three step recursion relation. In our present normalization this recursion relation is given by

$$xP_k(x) = -r_k \left[P_{k+1}(x) - P_k(x) - Q_k(P_k(x) - P_{k-1}(x)) \right], \tag{2}$$

where

$$r_k = -\frac{p_k}{p_{k+1}} \quad \text{and} \quad Q_k = \frac{h_k r_{k-1}}{r_k h_{k-1}}. \tag{3}$$

Here, the coefficient of x^k in $P_k(x)$ is denoted by p_k. The fractions r_k and the normalizations h_k can be determined from the relations

$$\int_0^\infty dx n V'(x) e^{-nV(x)} P_k(x) P_k(x) = -\int_0^\infty dx \frac{d}{dx}\left[e^{-nV(x)} P_k(x) P_k(x) \right] = 1, \tag{4}$$

$$\int_0^\infty dx n V'(x) e^{-nV(x)} x P_k(x) P_k(x) - h_k - 2k h_k = -\int_0^\infty dx \frac{d}{dx}\left[e^{-nV(x)} x P_k(x) P_k(x) \right] = 0. \tag{5}$$

For simplicity we consider the potential

$$V(x) = a_1 x + \frac{a_2}{2} x^2. \tag{6}$$

Then these relations reduce to

$$na_1 h_k + na_2 r_k h_k (1+Q_k) = 1, \tag{7}$$

$$na_1 r_k h_k (1+Q_k) + na_2 r_k^2 \left(h_{k+1} + h_k (1+Q_k)^2 + h_{k-1} Q_k^2 \right) - h_k - 2k h_k = 0. \tag{8}$$

In order to proceed we take the continuum limit of these relations, i.e. $k \to \infty$ and $n \to \infty$ at fixed $k/n = t$. With

$$\begin{aligned} r_k &\to r(t), \\ h_k &\to \frac{1}{n} h(t), \end{aligned} \tag{9}$$

to leading order in $1/n$, these recursion relations can be rewritten as

$$a_1 + 2a_2 r(t) = \frac{1}{h(t)}, \tag{10}$$

$$a_1 r(t) + 3a_2 r^2(t) = t. \tag{11}$$

The functions $r(t)$ and $h(t)$ are given by the solution of these equations. However, we do not need the explicit solution. A more useful property is that they satisfy a differential equation that does not depend on a_1 and a_2:

$$r'(t)h(t) - 2h'(t)r(t) = h^2(t). \tag{12}$$

Remarkably, as was shown by Akemann, Damgaard, Magnea and Nishigaki [1], this relation is valid for any polynomial potential. This relation is the essential ingredient in reducing the continuum limit of the recursion relation (2) to a Bessel equation.

To leading order in $1/n$ the r.h.s. of the recursion relation (2) is zero. We have to collect the terms of subleading order. In the continuum limit we may write

$$\begin{aligned} r_{k-1} &= r(t) - \frac{1}{n} r'(t), \\ h_{k-1} &= \frac{1}{n} h(t) - \frac{1}{n^2} h'(t), \\ P_{k-1}(x) &= P(t,x) - \frac{1}{n}\frac{d}{dt} P(t,x), \quad \text{etc..} \end{aligned} \tag{13}$$

This results in the differential equation

$$n^2 x P(t,x) = -h(t) \frac{d}{dt} \frac{r(t)}{h(t)} \frac{d}{dt} P(t,x). \tag{14}$$

We observe that the continuum limit of the recursion relation exists if we take at the same time the microscopic limit with fixed $n\sqrt{x}$. If we introduce the new variables

$$\begin{aligned} v &= n\lambda = n\sqrt{x}, \\ u(t) &= \frac{2\sqrt{r(t)}}{h(t)} = \int_0^t \frac{dt'}{\sqrt{r(t')}} \end{aligned} \tag{15}$$

the differential equation reduces to the Bessel equation

$$\frac{1}{u}\frac{d}{du}u\frac{d}{du}P(u,v^2) + v^2 P(u,v^2) = 0. \tag{16}$$

Here, and below we omit the argument t of $u(t)$. The general solution of this Bessel equation is given by

$$P(u,v^2) = AJ_0(uv) + BN_0(uv). \tag{17}$$

The constants are determined by the boundary condition that the orthogonal polynomials are normalized as $P_n(0) = 1$. This results in $A = 1$ and $B = 0$.

As discussed in lecture 3, the average spectral density follows from orthogonal polynomial kernel,

$$\rho(\lambda) = 2\lambda e^{-nV(\lambda^2)} \sum_{k=0}^{n-1} \frac{1}{h_k} P_k(\lambda^2) P_k(\lambda^2). \tag{18}$$

The factor 2λ results from the Jacobian of the transformation $x = \lambda^2$. We take the continuum limit of this relation and write the sum as an integral over $t = k/n$. In the microscopic limit, with $n\lambda$ fixed in the thermodynamic limit, the orthogonal polynomials can be replaced by their asymptotic limit (17). Taking into account that in this limit $\exp -nV(\lambda^2) \to 1$, we find

$$\rho(\lambda) = 2n^2 \int_0^1 \frac{dt}{h(t)} J_0^2(u(t)n\lambda). \tag{19}$$

From the definition of $u(t)$ it follows that

$$\frac{dt}{h(t)} = \frac{udu}{2}. \tag{20}$$

If we use the elementary result that

$$\int_0^1 udu J_0(uz) = \frac{1}{2}(J_0^2(z) + J_1^2(z)) \tag{21}$$

we find the following result for the spectral density

$$\rho(\lambda) = \frac{1}{2}\lambda n^2 u^2(1) \left(J_0^2(u(1)n\lambda) + J_1^2(u(1)n\lambda) \right). \tag{22}$$

For $n\lambda \to \infty$, but $\lambda \to 0$ the microscopic spectral density should coincide with $\rho(0)$. Using the asymptotic results for the Bessel functions as in previous lecture, we obtain that $u(1)$ can be identified as

$$u(1) = \frac{\pi\rho(0)}{n}. \tag{23}$$

For the microscopic spectral density defined by (with $z = N\Sigma\lambda = \pi\rho(0)\lambda$, notice that $N = 2n$)

$$\rho_S(z) = \lim_{N\to\infty} \frac{1}{N\Sigma}\rho(\frac{z}{N\Sigma}) = \lim_{n\to\infty} \frac{1}{\pi\rho(0)}\rho(\frac{z}{\pi\rho(0)}), \tag{24}$$

we obtain

$$\rho_s(z) = \frac{z}{2}(J_0^2(z) + J_1^2(z)), \tag{25}$$

in agreement with the result for the Gaussian chiral ensemble derived in the previous lecture. This result is independent of the specific shape of the random matrix potential. This proves its universality.

References

[1] P. Damgaard, U. Magnea and S. Nishigaki, Nucl. Phys. **B 487[FS]** (1997) 721.

[2] J.J.M. Verbaarschot, *Lectures given at NATO Advanced Study Institute on Confinement, Duality and Nonperturbative Aspects of QCD*, Cambridge, 1997, hep-th/9710114.