5 Analyticity and the Spectral Density for an Arbitrary Potential

In this lecture we study the average spectral density for a ensemble of Hermitian matrices with probability density given by

$$P(H)DH = e^{-N\mathrm{Tr}V(H)}DH, \tag{1}$$

where, for technical reasons, we restrict $V(H)$ to an arbitrary even polynomial in H. The joint eigenvalue density is given by

$$\rho(\lambda_1, \cdots, \lambda_N) = e^{-\sum_k V(\lambda_k) + \frac{1}{2}\sum_{k\neq l}\log|\lambda_k - \lambda_l|^2}. \tag{2}$$

The eigenvalues can be interpreted as the coordinates of particles interacting via a Coulomb-like potential in a confining harmonic well. For large N, the average position of the particles follows from a saddle-point approximation,

$$\partial_{\lambda_k} V = 2\sum_{l\neq k} \frac{1}{\lambda_k - \lambda_l}. \tag{3}$$

Replacing the sum by an integral we find an integral equation for the average spectral density

$$V(\lambda) = P\int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'}, \tag{4}$$

where $P\int$ denotes a principal value integral. This integral equation can be solved by means of the theory of singular integral equations. It can be shown that the general solution for the average spectral density is given by

$$\rho(\lambda) = \frac{2}{\pi^2} P\int_0^a \frac{tdt}{t^2-\lambda^2}\frac{dV}{dt}\left(\frac{a^2-\lambda^2}{a^2-t^2}\right)^{1/2}. \tag{5}$$

The support of the spectrum is given by the interval $[-a, a]$. The value of a follows from the normalization of $\rho(\lambda)$. For example, for the GUE, with $V'(t) = t$ we find $a = \sqrt{4N}$ and

$$\rho^{GUE} = \frac{1}{2\pi}\sqrt{a^2 - \lambda^2}. \tag{6}$$

Exercise 5.1. Show that the semicircular distribution is a solution of the integral equation (4) for potential $V(\lambda) = \lambda^2/2$.

However, the algebra of singular integral equations is quite tedious, and we follow a different route based entirely on analyticity. To this end we define the resolvent

$$G(z) = \frac{1}{N}\langle\sum_k \frac{1}{z-\lambda}\rangle. \tag{7}$$

The spectral density can be obtained by inversion,

$$\rho(\lambda) = N\frac{i}{2\pi}(G(\lambda + i\epsilon) - G(\lambda - i\epsilon)). \tag{8}$$

The eigenvalues of a Hermitian matrix are located on the real axis. For random matrices we expect that the support of the spectrum is compact, say inside the interval $[-a, a]$ (note that our probability distribution is an even function of the eigenvalues). We then find that, asymptotically for large z, the resolvent is given by

$$G(z) \sim \frac{1}{z} \quad \text{for} \quad z \to \infty. \tag{9}$$

A second property follows by inspection of the integral equation (4) and the relation

$$\frac{1}{\lambda + i\epsilon} = P\frac{1}{\lambda} - \pi i \delta(\lambda). \tag{10}$$

For $\lambda \in [-a, a]$ we then find

$$G(\lambda + i\epsilon) + G(\lambda - i\epsilon) = V'(\lambda). \tag{11}$$

A third property follows from the cut-structure of the resolvent in the complex z-plane. For a finite density of eigenvalues everywhere along the interval $[-a, a]$, because of the relation (8), the resolvent has a cut along this interval. For a smooth spectral density this is the only singularity. The general form of the average resolvent is thus given by

$$G(z) = A(z) + B(z)\sqrt{z^2 - a^2}, \tag{12}$$

where $A(z)$ and $B(z)$ are regular analytic functions. For $z \in [-a, a]$, the value of $A(z)$ follows from (11). Because $A(z)$ is a analytic function, have that for all z in the complex plane

$$A(z) = \frac{1}{2}V'(z). \tag{13}$$

Before writing down the general solution, let us first study two simple cases. The simplest case is the GUE with $V'(z) = z$. Then

$$G(z) = \frac{z}{2} + (b_0 + b_1 z^2 + \cdots)z(1 - \frac{1}{2}\frac{a^2}{z^2} - \frac{1}{8}\frac{a^4}{z^4} + \cdots). \tag{14}$$

We are looking a solution consistent with our requirements for $G(z)$. This should be the unique solution of our problem. Asymptotically for large z, we have that $G(z) \sim 1/z$. However, only the term proportional to b_0 in $B(z)$ can be canceled by $z/2$. We thus find that $b_k = 0$ for $k \geq 1$ and $b_0 = -1/2$. From the coefficient of $1/z$ we find the relation

$$-\frac{b_0}{2a^2} = 1. \tag{15}$$

With the value of $b_0 = -1/2$, this results in $a = 2$. Our final result for the resolvent of the GUE is then given by

$$G(z) = \frac{z}{2} - \frac{1}{2}\sqrt{z^2 - 4} \tag{16}$$

resulting in the spectral density is then given by (see (8))

$$\rho(\lambda) = \frac{N}{2\pi}\sqrt{4 - z^2}. \tag{17}$$

Let us now study the case

$$V'(z) = z + d_2 z^3. \tag{18}$$

We again consider the expansion of the resolvent for large z. In this case the term proportional to b_1 can be cancelled by the $d_2 z^3$ term, but $b_k = 0$ for $k \geq 2$. From the asymptotic $1/z$ dependence of $G(z)$ we find the following three relations for b_0, b_1 and a,

$$b_0 - \frac{b_1 a^2}{2} + \frac{1}{2} = 0, \tag{19}$$

$$b_1 + \frac{d_2}{2} = 0, \tag{20}$$

$$-\frac{b_1 a^4}{8} = 1. \tag{21}$$

This results in the average resolvent

$$G(z) = \frac{1}{2}(z + d_2 z^3) - (\frac{1}{2} - \frac{d_2 a^2}{4} + \frac{d_2}{2} z^2)\sqrt{z^2 - \frac{4}{\sqrt{d_2}}}. \tag{22}$$

By now it should be clear how to solve the general case with potential given by

$$V'(z) = \sum_{k=1}^{p} d_k z^{2k-1}. \tag{23}$$

An asymptotic $1/z$ behavior is only possible if

$$B(z) = \sum_{k=0}^{p-1} b_k z^{2k}. \tag{24}$$

The expansion of the square root is given by

$$\sqrt{z^2 - a^2} = z(1 - \sum_{k=1}^{\infty} a_k \frac{a^{2k}}{z^{2k}}). \tag{25}$$

with

$$a_k = \frac{(2k-2)!}{(k-1)!k!2^{2k-1}}. \tag{26}$$

The relation between the b_k and the d_k is given by

$$d_k = 2\sum_{r=0}^{p-1} b_r a_{r-k+1} a^{2(r-k+1)}, \tag{27}$$

where by definition $a_0 = -1$. The final explicit answer is then obtained by inverting this relation whereas the value of a follows from the coefficient of $1/z$,

$$\sum_{k=1}^{p-1} b_k a_{k+1} a^{2k+2} + 1 = 0. \tag{28}$$

The inversion can be performed by menas of the following inversion relation [4]. If

$$a_n = \sum_{k=0}^{n} \frac{1}{k+1} \left(\begin{array}{c} 2k \\ k \end{array} \right) b_{n-k}, \tag{29}$$

then

$$b_n = a_n - \sum_{k=1}^{n} \frac{1}{k} \left(\begin{array}{c} 2k \\ k \end{array} \right) a_{n-k}. \tag{30}$$

Excercise 5.2. Invert the relation (27).

References

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