6 Universality in RMT: orthogonal polynomials

In this lecture we derive an asymptotic limit for orthogonal polynomials valid in the region well inbetween the smallest and the largest zero. We will derive a formula first proposed by Brézin and Zee [?] and follow the derivation given by Eynard [8]. This derivation exploits analyticity arguments similar to the ones discussed in previous chapter for the derivation of the average spectral density.

6.1 Partition function for orthogonal polynomials

The proof of Eynard is based on the following general expression for orthogonal polynomials

$$p_n(x) = \sqrt{\frac{n+1}{Z_n Z_{n+1}}} \int dM e^{-N\mathrm{tr}V(M)} \det(x-M), \tag{1}$$

where the integration is over $n \times n$ Hermitean matrices M and

$$Z_n = \int dM e^{-\mathrm{tr}NV(M)}. \tag{2}$$

Excercise 6.1. Prove this expression.

To find the asymptotics of the orthogonal polynomials, we study a slightly more general partition function,

$$Z(g,h,x) = \int dM e^{-n/g\mathrm{tr}\mathcal{V}(M)}, \tag{3}$$

where

$$\mathcal{V}(z) = V(z) - h\log(x-z). \tag{4}$$

In terms of the eigenvalues of the matrix M this partition function can be written as (up to a constant)

$$Z(g,h,x) = \int d\lambda_1 \cdots d\lambda_n \Delta^2(\{\lambda_k\}) e^{-n/g\sum_k [V(\lambda_k) - h\log(\lambda_k - x)]}, \tag{5}$$

where $\Delta(\{\lambda_k\})$ is the Vandermonde determinant. We wish to evaluate this partition function in the limit $n \to \infty$. In this limit the integral over the eigenvalues can be approximated by a saddle-point approximation. Then the eigenvalues satisfy the equation

$$\frac{1}{g}\mathcal{V}'(\lambda_k) = \frac{2}{n}\sum_{l\neq k}\frac{1}{\lambda_k - \lambda_l}. \tag{6}$$

It is useful to introduce the resolvent

$$\omega(g,h,z) = \frac{1}{n}\langle \mathrm{tr}\frac{1}{z-M}\rangle. \tag{7}$$

To leading order in $1/n$ the resolvent follows from the saddle-point equation of the eigenvalues. However, we will evaluate the $1/n$ corrections to the resolvent as well.

Since the resolvent is given by

$$\omega(z) = \frac{1}{n}\frac{\partial}{\partial z}\log Z(g,h,z), \tag{8}$$

the partition function can be obtained by integrating this identity. However, we will need the $1/n$ corrections to the saddle-point result for the resolvent to derive the asymptotic result for the orthogonal polynomials. It turns out that they follow from the g and h dependence of the average resolvent.

6.2 Analyticity and $1/n$ corrections to the resolvent

As we have seen in previous lecture, the saddle-point equation for the eigenvalues can be expressed in terms of the average resolvent

$$\frac{1}{g}\mathcal{V}'(\lambda) = \omega(g,h,\lambda+i\epsilon) + \omega(g,h,\lambda-i\epsilon) \tag{9}$$

for λ inside the support of the spectrum of the random matrix. From this equation we can immediately derive equations for the derivatives of ω

$$\partial_g(g\omega(g,h,\lambda+i\epsilon)) + \partial_g(g\omega(g,h,\lambda-i\epsilon)) = \partial_g\mathcal{V}'(\lambda), \tag{10}$$

$$\partial_h(g\omega(g,h,\lambda+i\epsilon)) + \partial_h(g\omega(g,h,\lambda-i\epsilon)) = \partial_h\mathcal{V}'(\lambda) = \frac{1}{x-z} \tag{11}$$

$$\tag{12}$$

To first order in $g-1$ and h the average resolvent can be expanded as

$$g\omega(g,h,z) = \omega(1,0,z) + (g-1)\Omega_g + h\Omega_h. \tag{13}$$

The term $\omega_0(z) \equiv \omega(1,0,z)$ was determined in previous lecture. It is given by

$$\omega_0(x+i\epsilon) = \frac{1}{2}V'(x) - \pi i\rho(x)/n. \tag{14}$$

Next we determine Ω_g and Ω_h from analyticity arguments only. Let us first determine Ω_g. Because of the derivative of the endpoints of the spectrum with respect to g, the behavior of Ω_g near $z = \pm a$ is given by $\Omega_g \sim 1/\sqrt{z^2-a^2}$. The general from of Ω_g is thus given by

$$\Omega_g = A(z) + \frac{B(z)}{\sqrt{z^2-a^2}}, \tag{15}$$

where $A(z)$ and $B(z)$ are regular functions. Because of (10) we find that $A(z) = 0$. By differentiating the partition function, we find that for large z the derivative $\Omega_g \sim 1/z$. Because $g = n/N$ we obtain from (13) that $\Omega_g \to \frac{1}{z}$ for $z \to \infty$ (note that $\omega(1,0,z) \to 1/z$ for $z \to \infty$). This is only possible if $B(z) = 1$, and our result for Ω_g is given by

$$\Omega_g = \frac{1}{\sqrt{z^2-a^2}}. \tag{16}$$

Next we use similar arguments to determine Ω_h. By differentiating the partition function, we find that for large z the derivative $\Omega_h \sim 1/z^2$. By differentiating the endpoints of the spectrum with respect to h we find that also in this case the general form of Ω_h is given by

$$\Omega_h = A(z) + \frac{B(z)}{\sqrt{z^2-a^2}}, \tag{17}$$

where $A(z)$ and $B(z)$ are analytic functions of z. From (11) we find that $A(z) = 1/(2(x-z))$. Asymptotically for large z, we have that $\Omega_h \sim 1/z^2$. This follows from the asymptotic g/z-dependence of $g\omega(g,h,z)$. Therefore, $B(z) = b_0 + b_1/(x-z)$. In order to cancel the $1/z$ dependence of $A(z)$ we require that $b_0 = 1/2$. On the real axis we want a cut singularity so that the $1/z-x$ pole has to be cancelled. This is only possible if $b_1 = \sqrt{x^2-a^2}/2$. Our final result for Ω_h is thus given by

$$\Omega_h = \frac{1}{2}\frac{1}{\sqrt{z^2-a^2}}\left(1 - \frac{\sqrt{z^2-a^2}-\sqrt{x^2-a^2}}{z-x}\right). \tag{18}$$

6.3 Asymptotic result

To derive the asymptotic result for the orthogonal polynomials, we return to the original partition function. It is related to the resolvent by

$$\log Z = \text{constant} + n\int_0^x \omega(\frac{n}{N},\frac{1}{n},x')dx'. \tag{19}$$

The polynomials are then given by

$$P_n(x) \sim Z \sim e^{\int_0^x \omega(\frac{n}{N},\frac{1}{N},x')dx'}. \tag{20}$$

The asymptotic result for the resolvent is given by

$$\omega(\frac{n}{N},\frac{1}{N},x+i\epsilon) = \frac{N}{n}(\omega_0 + (\frac{n}{N}-1)\Omega_g + \frac{1}{N}\Omega_h), \tag{21}$$

where ω_0, Ω_g and Ω_h are evaluated at $z = x+i\epsilon$. In this limit

$$\Omega_h = \frac{1}{2}\frac{1}{\sqrt{z^2-a^2}} - \frac{1}{2}\frac{d}{dx}\log\sqrt{x^2-a^2} \tag{22}$$

To integrate ω we introduce

$$\zeta(x) = -\frac{\pi}{n}\int_a^x \rho(x')dx'. \tag{23}$$

The quantity Ω_g can be simply integratedby changing variables $x = a\cos\phi$. This results in

$$\int_a^x \Omega_g(x')dx' = \phi(x). \tag{24}$$

For a polynomial we have that

$$P_n(x) = \frac{1}{2}(P_n(x+i\epsilon) + P_n(x-i\epsilon)). \tag{25}$$

Finally combining everything we find

$$P_n(x) \sim e^{NV(x)/2}\frac{1}{(a^2-x^2)^{1/4}}\cos(N\zeta(x) - (N-n+\frac{1}{2})\phi(x) - \frac{\pi}{4}), \tag{26}$$

where the phase $\pi/4$ results from writing $\sqrt{(x+i\epsilon)^2-a^2} = i\sqrt{a^2-x^2}$ in the argument of the logarithm in Ω_h.

6.4 Average spectral density

The average spectral density is given by

$$\rho(x) = c^2 e^{-NV(x)} \sum_{n=0}^{N-1} P_n(x) P_n(x). \tag{27}$$

The sum over n can be performed by means of the Christoffel-Darboux formula,

$$\sum_{n}^{N-1} P_n^2(x) = P_N(x) P'_{N-1}(x) - P'_N(x) P_{N-1}(x). \tag{28}$$

By using a goniometric formula we then find

$$\rho(x) = -c^2 N \zeta'(x), \tag{29}$$

from which it follows that $c = 1/\sqrt{\pi}$.
Excercise 6.1. Derive the microscopic limit of the two-point correlation function in the bulk of the spectrum using the asymptotic result for the orthogonal polynomials.

References

[1] E. Br/'ezin and A. Zee, Nucl. Phys. **B 453** [FS] (1995) 531.

[2] B. Eynard, Nucl. Phys. **B 506** [FS] (1997) 633.