7 Grassmann integration

In this lecture we introduce Grassmann variables and introduce integration over Grassmann variables. The basic objects that we will consider are vectors consisting out of m commuting components and n anti-commuting components. Such vectors will be called graded vectors or super-vectors, and the matrices acting on these vectors will be called $(m|n)$ graded matrices or super-matrices. Our final goal will be to establish an integration measure on manifolds based on this graded structure.

7.1 Definitions

Grassmann variables are anti-commuting variables

$$\{\chi_k, \chi_l\} = 0, \tag{1}$$

which are necessarily nilpotent, i.e. $\chi_k^2 = 0$. One can introduce two types of conjugation, namely, conjugation of the first kind with the property that

$$\chi^{**} = \chi, \tag{2}$$

and conjugation of the second kind with the property

$$\chi^{**} = -\chi. \tag{3}$$

Below it will become clear that it is often advantageous to use conjugation of the second kind.

The integration over Grassmann variables is defined by

$$\int d\chi = 0, \qquad \int \chi d\chi = 1, \tag{4}$$

and thus more resembles like a differentiation than an integration. As an example, consider the integral

$$\int d\chi d\chi^* e^{\chi a \chi^*} = \int d\chi d\chi^* (1 + \chi a \chi^*) = -a. \tag{5}$$

Notice that the differentials $d\chi$ are also anti-commuting elements.

Next let us make a change of integration variables, $\xi = a\ chi$. Then

$$1 = \int \xi d\xi = \int a\chi d(a\chi), \tag{6}$$

and we find the relation

$$d(a\chi) = \frac{1}{a} d\chi. \tag{7}$$

For a general transformation given by $\xi_k = A_{kl}\chi_l$ we have that

$$d\xi_1 \cdots c\xi_n = \frac{1}{\det A} d\chi_1 \cdots \chi_n. \tag{8}$$

Exercise 7.1. Prove this relation.

An important class of integrals are Gaussian Grassmannian integrals. Let us consider

$$I \equiv \int d\chi_1 \cdots d\chi_n d\chi_1^* \cdots d\chi_n e^{\chi_k^* A_{kl} \chi_l}. \tag{9}$$

Because of the nilpotency and the definition of the Grassmann integration we obtain

$$I \equiv \int d\chi_1 \cdots d\chi_n d\chi_1^* \cdots d\chi_n \frac{1}{n!} (\chi_k^* A_{kl} \chi_l)^n. \tag{10}$$

By renaming the integration variables one easily finds that $n!$ of the terms contribution to the integral are the same. We thus find

$$I = \int d\chi_1 \cdots d\chi_n d\chi_1^* \cdots d\chi_n^* \sum_\pi \chi_1^* A_{1\pi(1)} \chi_{\pi(1)} \cdots \chi_n^* A_{n\pi(n)} \chi_{\pi(n)}. \tag{11}$$

where the sum is over all permutations of $1, \cdots, n$. By rearranging the differential it easily follows that the result of Grassmann integrations is exactly the sign of the permutation, and

$$I = \det A. \tag{12}$$

This should be compared to a bosonic integral

$$\frac{1}{\pi} \int d\phi_1 \cdots d\phi_n d\phi_1^* \cdots d\phi_n^* e^{-\phi_k^* A_{kl} \phi_l} = \frac{1}{\det A}. \tag{13}$$

Exercise 7.2. Prove this result.

One essential difference between fermionic and bosonic integrals is that fermionic integrals are always convergent, whereas bosonic integrals only exist for specific integration contours.

7.2 Graded vectors and Graded Matrices

An $(m|n)$ graded vector is defined by

$$\phi_k \equiv \begin{pmatrix} S_1 \\ \vdots \\ S_m \\ \chi_1 \\ \vdots \\ \chi_n \end{pmatrix}. \tag{14}$$

The set of matrices acting on these vectors is given by

$$M = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}, \tag{15}$$

where a and b are matrices with commuting entries and ρ and σ are matrices with anti-commuting entries. The block structure is such that a is an $m \times m$ matrix and b is an $n \times n$ matrix.

The trace of a graded matrix is defined by

$$\mathrm{Trg} M = \mathrm{Tr} a - \mathrm{Tr} b. \tag{16}$$

A graded trace of a product of graded matrices is invariant under cyclic permutations.

The graded determinant is defined by the relation

$$\mathrm{detg} M = \exp \mathrm{Trg} \log M = \exp \mathrm{Trg} \log \left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} [1 + \begin{pmatrix} 0 & a^{-1}\sigma \\ b^{-1}\rho & 0 \end{pmatrix}] \right]. \tag{17}$$

The logarithm can be expanded in a power series and only the even powers contribute to the graded trace. Resumming the power series again into a logarithm, one finds

$$\mathrm{detg} M = \frac{\det(a - \sigma b^{-1}\rho)}{\det b}. \tag{18}$$

Exercise 7.3. Show that $\mathrm{detg}(AB) = \mathrm{detg}\, A \mathrm{detg} B$.

The transpose of the matrix M is defined by

$$M^T = \begin{pmatrix} a^T & \rho^T \\ -\sigma^T & B^T \end{pmatrix}. \tag{19}$$

Because of the additional minus sign we have the relation

$$(M_1 M_2)^T = M_2^T M_1^T. \tag{20}$$

For a real scalar product the transpose satisfies the usual relation

$$(phi_1, M\phi_2) = (M^T \phi_1, \phi_2). \tag{21}$$

The Hermitian conjugate is defined in usual way

$$M^\dagger = M^{*T}. \tag{22}$$

A super-matrix is Hermitian is $M^\dagger = M$.

The eigenvalues of a supermatrix are defined by

$$M\phi_k = \lambda_k \phi_k. \tag{23}$$

For two different eigenvectors we have

$$(\phi^k, M\phi^l) = (M^\dagger \phi^k, \phi^l), \tag{24}$$

where we have used the conjugation of the second kind. For a super-Hermitean matrix we then find

$$(\lambda_k - \lambda_l)(\phi_k, \phi_l) = 0, \tag{25}$$

so that the eigenvectors corresponding to different eigenvalues are orthogonal. As a consequence, a super-Hermitian matrix can be diagonalized by a super-unitary matrix, where a super-unitary matrix is defined in the usual way

$$U^\dagger U = 1. \tag{26}$$

Next, let us calculate the following Gaussian graded integral

$$I \equiv \frac{1}{\pi} \int d\phi d\phi^* e^{-\phi_k * M_{kl} \phi_l}, \tag{27}$$

where, for reasons of convergence, the matrix M is a super-Hermitian matrix. In the block notation introduced earlier in this chapter, the exponent can be written as

$$\begin{aligned} \phi_k * M_{kl} \phi_l &= S^\dagger a S + S^\dagger \sigma \chi + \chi^\dagger \rho S + \chi^\dagger b \chi \\ &= (S^\dagger + \chi^\dagger \rho a^{-1}) a (S + a^{-1} \sigma \chi) + \chi^\dagger (b - \rho a^{-1} \sigma) \chi. \end{aligned} \tag{28}$$

After shifting the bosonic integration variables, the integral factorizes in a bosonic and fermionic piece which we know how to do. The result is

$$I = \frac{\det(\rho a^{-1} \sigma - b)}{\det a} = \frac{\det b}{\det(a - \sigma b^{-1} \rho)} = \frac{1}{\mathrm{detg} M}. \tag{29}$$

The second identity follows by writing the determinant as an exponent of the trace of a logarithm, and perfrom a cyclic permutation of the graded trace. The minus sign resulting in the inverse determinant result from the anti-commutation of the Grassmann variables.

Now we can understand why the graded trace was introduced as in (16). It results in graded Gaussian integral that is given by an inverse graded determinant. For example, consider the case with zero Grassmannian blocks in the supermatrix.

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