# 8 Integration Theorems

If an integrand is invariant under super-unitary transformations, there exist a number of powerful integration theorems. In character they are comparable to a complex contour integration.

## 8.1 The Parisi-Sourlas reduction

Let us first consider the simplest case involving the graded vectors

$$p = \begin{pmatrix} a \\ \theta \end{pmatrix}, \qquad p^\dagger = (a^*, \theta^*) \,. \tag{1}$$

Let $F(p, p^\dagger)$ be an invariant function, i.e.

$$F(Up, U^\dagger P^\dagger) = F(p, p^\dagger), \tag{2}$$

for an arbitrary super-unitary transformation $U$. In this case the integration theorem is

$$\frac{1}{\pi} \int dada^* d\theta d\theta^* F(p, p^\dagger) = F(0, 0). \tag{3}$$

To prove this theorem, we expand $F$ in powers of the Grassmann variables

$$F(p, p^\dagger) = F_{00} + F_{01}\theta^* + F_{10}\theta + F_{11}\theta^*\theta. \tag{4}$$

If we choose

$$u = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \tag{5}$$

then the r.h.s. is only invariant if $F_{01} = F_{10} = 0$. By performing an infinitesimal transformation

$$u = \begin{pmatrix} 1 & \alpha \\ \alpha^* & 1 \end{pmatrix}, \tag{6}$$

we find that

$$F_{11} = \frac{1}{a} \partial_{a^*} F_{00}. \tag{7}$$

*Exercise 8.1.* Show that $F_{00}$ is a function of $aa^*$ After performing the Grassmann integration, our integral is thus given by

$$\frac{1}{\pi} \int dada^* \frac{1}{a} \partial_{a^*} F_{00}. \tag{8}$$

In polar coordinates the integral can be rewritten as

$$\frac{1}{2\pi} \int \rho d\rho d\phi \frac{1}{\rho} \partial_\rho F(\rho). \tag{9}$$

For a function that is well-behaved at infinity we then find that the integral is given by $F(0, 0)$.

This theorem can be extended to supervectors with an arbitrary, but equal number of commuting and anticommuting components. We leave it up to the reader to formulate this theorem.

## 8.2 The Wegner Theorem

Next we consider an integration theorem for integrals over invariant functions of a supermatrix. The original lectures of Wegner were worked out in a paper by Constantinescu and Groote. In this case the invariant function satisfies the relation

$$F(Q) = F(UQU^{-1}), \tag{10}$$

where $U$ is a super-unitary matrix. The general form of an invariant function is thus given by

$$F(Q) = \sum_k a_k \mathrm{Trg} Q^k. \tag{11}$$

Therefore, automatically, $F$ is also invariant under general super-linear transformations. The theorem states that

$$\int DQF(Q) = F(\lambda \mathbf{1}), \tag{12}$$

where $\lambda$ is an arbitrary constant. Notice that this constant is not determined for an invariant function. We first prove this theorem for the simplest case of $Q$ a $(1|1)$ matrix. Then any function of the $(1|1)$ graded matrix

$$Q = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \tag{13}$$

can be expanded as

$$F(Q) = F_{00} + \sigma F_{01} + \rho F_{10} + \sigma\rho F_{11}. \tag{14}$$

If we choose

$$U = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \tag{15}$$

Then,

$$UQU^{-1} = \begin{pmatrix} a & \sigma e^{i\phi - i\theta} \\ \rho e^{i\theta - i\phi} & b \end{pmatrix}. \tag{16}$$

Invariance of $F(Q)$ then requires that $F_{10} = F_{01} = 0$ in the expansion (14). Next consider invariance under

$$U = \begin{pmatrix} 1 & \omega \\ \zeta & 1 \end{pmatrix}. \tag{17}$$

Then to first order in $\omega$ and $\zeta$

$$UQU^{-1} = \begin{pmatrix} a + \omega\rho + \zeta\sigma & \omega(b-a) + \sigma \\ \zeta(a-b) + \rho & b + \omega\rho + \zeta\sigma \end{pmatrix}. \tag{18}$$

From the invariance of $F$ we then find

$$(\omega\rho + \zeta\sigma)(\partial_a F_{00} + \partial_b F_{00}) + (b-a)(\omega\rho + \zeta\sigma)F_{11} = 0. \tag{19}$$

This result is

$$F_{11} = \frac{\partial_a F_{00} + \partial_b F_{00}}{a-b}. \tag{20}$$

*Example.* Consider the invariant function $F(Q) = \mathrm{Trg} Q^2$. In components we obtain $F(Q) = a^2 - b^2 + 2\sigma\rho$, which is in agreement with the above general result.

Let us perform the integral. After doing the Grassmann integrations we find

$$\int DQ F(Q) = \frac{1}{\pi i} \int dadb \frac{\partial_a F_{00} + \partial_b F_{00}}{a-b}. \tag{21}$$

This integral is not well defined if $a$ and $b$ are along the same path in the complex plane. For definiteness let us take $a$ along the real axis and $b$ along the imaginary axis. In polar coordinates

$$a = \rho \cos\phi, \tag{22}$$
$$b = i\rho \sin\phi. \tag{23}$$

Then

$$\partial_a F_{00} + \partial_b F_{00} = (\frac{a-b}{\rho}\partial_\rho + \frac{a-b}{i\rho^2}\partial_\phi) F_{00}. \tag{24}$$

The integral over $\phi$ of the second term gives zero provided that the point $\rho = 0$ is suitably regularized. The integral over $\rho$ of the first term is the integral of a total derivative. For well-behaved functions $F$ we thus find

$$\int DQ F(Q) = F(0). \tag{25}$$

An inductive proof for the general case of this theorem was given by Wegner. He considered the case of an arbitrary $(m|n)$ matrix $Q$. In that case an integral over an $(m-n) \times (m-n)$ ordinary matrix remains (for $m > n$). However, for $m = n$, the theorem is exactly as in (25) but with $Q$ an $(m|n)$ super-matrix.

The proof of the theorem starts from the observation that an arbitrary $(m|n)$ matrix can be decomposed into the $(1|1)$ super-matrix

$$\bar{Q} = \begin{pmatrix} Q_{11} & Q_{1,n+m} \\ Q_{n+m,1} & Q_{n+m,n+m} \end{pmatrix}, \tag{26}$$

the $(1|1)$ super-vectors

$$P_\alpha = \begin{pmatrix} Q_{1\alpha} \\ Q_{n+m\alpha} \end{pmatrix}, \qquad \alpha = 2, \cdots n+m-1, \tag{27}$$

and

$$\bar{P}_\alpha = (Q_{\alpha 1}, Q_{\alpha n+m}), \qquad \alpha = 2, \cdots n+m-1. \tag{28}$$

The remaining matrix elements will be denoted by $Q_R$. We can thus write

$$F(Q) = \bar{F}(\bar{Q}, P, \bar{P}, Q_R). \tag{29}$$

We now consider the following invariance

$$\bar{F}(U\bar{Q}U^{-1}, UP, \bar{P}U^{-1}, Q_R) = \bar{F}(\bar{Q}, P, \bar{P}, Q_R). \tag{30}$$

and define

$$I(P, \bar{P}, Q_R) = \int D\bar{Q}\bar{F}(\bar{Q}, P, \bar{P}, Q_R). \tag{31}$$

By using the invariance of the measure one easily obtains that

$$I(UP, \bar{P}U^{-1}, Q_R) = I(P, \bar{P}, Q_R). \tag{32}$$

We thus can apply the Parisi-Sourlas theorem to the $P$ and $\bar{P}$ integrations. Next we can apply Wegners theorem for $(1|1)$ matrices to the $\bar{Q}$ integrations. What remains is an integral over the smaller super-matrix $Q_R$, thereby proving the general form of the theorem.

### 8.3 The Efetov-Zirnbauer theorem

In general one is interested in super-integrals of a non-supersymmetric integrand. In order to study this type of integrals, let us first study the $(1|1)$ case given by

$$\int DQ(a_{00}(Q) + a_{10}(Q)\sigma + a_{01}(Q)\rho + a_{11}(Q)\sigma\rho), \tag{33}$$

where $Q$ is parametrized as in (13) and $a_{ij}(Q)$ are invariant functions. Because of the general decomposition of invariant functions, we immediately find that the integrals of the second and third terms are zero. However, as was observed by Zirnbauer, there is another way of deriving this result.

The proof by Zirnbauer [5] is based on the invariance of the measure. If we apply the transformation (18) to the $Q$ variables of the integral of the second term we obtain

$$\int DQF(Q)a_{10}\sigma\rho = \int DQF(Q)a_{10}(\sigma + \omega(b-a))(\rho + \zeta(a-b)) \tag{34}$$

where we have applied the transformation (18) to the $Q$ variables. Since this equation is valid for arbitrary $\rho$ and $\zeta$, and we find that the second and third term in (33) integrate to zero.

To better appreciate this theorem let us diagonalize the matrix $Q$. The eigenvalues of a graded matrix with blockstructure (13) are given by the zeros and the poles of the secular equation

$$\mathrm{detg}(Q - \lambda) = \frac{\det(a - \lambda - \sigma(b-\lambda)^{-1}\rho)}{\det(b-\lambda)} = 0. \tag{35}$$

For example, consider the eigenvalues of a diagonal matrix. The poles of the secular equation can be obtained from the equation

$$\mathrm{detg}\frac{1}{(Q-\lambda)} = \frac{\det(b - \lambda - \rho(a-\lambda)^{-1}\sigma)}{\det(a-\lambda)} = 0. \tag{36}$$

The ordinary parts of the eigenvalues are obtained by putting $\sigma$ and $\rho$ equal to zero and are thus given by the eigenvalues of $a$ and $b$. The eigenvalues with ordinary part equal to one of the eigenvalues of $a$ are given by the solutions of eq. (35). The eigenvalues with ordinary part equal to one of the eigenvalues of $b$ are given by the solutions of eq. (36). Notice that eq. (35) is singular for $\lambda$ equal to one of the eigenvalues of $b$.

*Exercise 8.2.* Show that the eigenvalues of a (1|1) super-matrix are given by

$$\lambda_1 = a - \frac{\sigma\rho}{a-b}, \tag{37}$$

$$\lambda_2 = b - \frac{\sigma\rho}{a-b}. \tag{38}$$

Now let us consider a more complicated integral involving an invariant function of $Q^\dagger Q$ with $Q$ the (1|1) supermatrix given in eq (18). The matrix $Q^d aggerQ$ is then given by

$$Q^d aggerQ = \begin{pmatrix} a^*a + \rho^*\rho & a^*\sigma + b\rho^* \\ -a\sigma^* + b^*\rho & b^*b - sigma^*sigma \end{pmatrix}. \tag{39}$$

The eigenvalues of this matrix are given by

$$\lambda_1 = a^*a + \rho^*\rho - \frac{a^*a\sigma^*\sigma + b^*b\rho^*\rho + a^*b^*\sigma\rho + ab\sigma^*\rho^*}{a^*a - b^*b} + \frac{(a^*a + b^*b)\sigma^\sigma + \rho^*\rho}{(a^*a - b^*b)^2}, \tag{40}$$

$$\lambda_2 = \lambda_1 - a^*a + b^*b - \sigma^*\sigma - \rho^*\rho. \tag{41}$$

A general invariant function of $Q^\dagger Q$ can therefore be decomposed as

$$F(Q^\dagger Q) = F_{00} + F_{11}\sigma\rho + F_{11}^*\sigma^*\rho^* + F_{20}\sigma^*\sigma + F_{02}\rho^*\rho + F_{22}\sigma^*\sigma\rho^*\rho. \tag{42}$$

The invariance is given by

$$Q \to UQV, \tag{43}$$

where $U$ and $V$ are super-unitary matrices.

Because of this decomposition the super-invariant integral of the product of an invariant function and an odd number of Grassmann variables is trivially zero. According to a theorem due to Zirnbauer a super-invariant integral over the product of an invariant function and two Grassmann variables also vanishes. To proof proceeds as in our previous example and consider the infinitesimal transformation

$$\begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \to \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix} \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \begin{pmatrix} 1 & \zeta \\ \sigma & 1 \end{pmatrix}. \tag{44}$$

The Grassmann variables $\sigma$ and $\rho$ transform as

$$\sigma \to \sigma + a\zeta + b\alpha + \rho\alpha\zeta, \tag{45}$$

$$\sigma^* \to \sigma^* - a\alpha - b\zeta - \rho\alpha\zeta, \tag{46}$$

$$\rho \to \rho + b\omega + a\beta + \beta\sigma\omega, \tag{47}$$

$$\rho^* \to \rho^* - b\beta - a\omega - \beta\sigma\omega. \tag{48}$$

If we apply this transformation to

$$\int dQ f(Q^\dagger Q)\sigma\sigma^*\rho\rho^*, \tag{49}$$

and use the invariance of $f(Q^d agger Q)$ and the invariance of the measure we find that the integral of the product of an invariant function and two Grassmann variables vanishes.

One might think that the same argument can be applied to the integral of an invariant function. Obviously, because of Wegner's theorem this is not true. The reason is that the integral (49) diverges in this case. Typically, the invariant measure diverges if two eigenvalues coincide. Below we will calculate the invariant measure for this case.

### 8.4 Efetov-Wegner terms

Let us study the following (1|1) super-invariant integral

$$I = \frac{1}{\pi} \int dadbd\sigma d\rho e^{-\mathrm{Trg} Q^\dagger Q}, \tag{50}$$

where

$$Q = \begin{pmatrix} a & \sigma \\ \rho & ib \end{pmatrix}. \tag{51}$$

The additional factor $i$ has been inserted so that the integral converges for the $b$-integration over the real axis. By expanding the Grassmann variabels and performing the Gaussian integral one immediately finds that $I = 2$. Next we choose the eigenvalues of $Q$ and $\sigma$ and $\rho$ as new interation variables. The Jacobian of this transformation is given by

$$J = \det \begin{pmatrix} 1 + \frac{\sigma\rho}{(a-ib)^2} & \frac{-i\sigma\rho}{(a-ib)^2} \\ \frac{\sigma\rho}{(a-ib)^2} & i - \frac{i\sigma\rho}{(a-ib)^2} \end{pmatrix} = i. \tag{52}$$

We thus find that our integral is given by

$$I = \int \frac{d\lambda_1 d\lambda_2 d\sigma d\rho}{\pi i} e^{-\lambda_1^2 + \lambda_2^2} = 0, \tag{53}$$

where the integral over $\lambda_2$ is along the imaginary axis. However, because of the Grassmann integration the result of the integral is zero. Obviously, something went wrong! The only possibility is at the point $a = ib$ where our varible tranformation becomes singular.

Let us do the integral by excluding an infinitesimal sphere around this point from the integration domain. This approach was followed by Haldane and Zirnbauer and by Gossiaux, Pluhar and Weidenmüller. In terms of the original variables this changes the integral by only an infinitesimal amount. Our integral is thus given by

$$I = \lim_{\epsilon \to 0} I_\epsilon \tag{54}$$

with

$$I_\epsilon = \int dQ e^{-\mathrm{Trg} Q^2} \theta(\sqrt{a^2 + b^2} - \epsilon). \tag{55}$$

In terms of our new integration variables we have that

$$\sqrt{a^2 + b^2} = \sqrt{\lambda_1^2 - \lambda_2^2} - \frac{\alpha\beta}{\sqrt{\lambda_1^2 - \lambda_2^2}}, \tag{56}$$

and our $\theta$ function is given by

$$\theta(\sqrt{a^2+b^2}-\epsilon) = \theta(\sqrt{\lambda_1^2-\lambda_2^2}-\epsilon) - \sigma\rho\frac{1}{\sqrt{\lambda_1^2-\lambda_2^2}}\delta(\sqrt{\lambda_1^2-\lambda_2^2}-\epsilon). \tag{57}$$

If we use polar coordinates for $\lambda_1$ and $\lambda_2/i$, the anomalous contribution to our integral can be written as

$$\frac{1}{\pi}\int \rho d\rho d\phi d\sigma d\rho \frac{1}{\rho}\delta(\rho-\epsilon)e^{-\rho^2}\rho\sigma = 2, \tag{58}$$

which is the correct answer for our Gaussian integral. In this case the non-anamalous piece is zero.

In general one is interested in non-superinvariant integrals of the following general type

$$I = \int dQ(f_0(Q) + rho\sigma f_1(Q)). \tag{59}$$

In terms of our new integration variables the integral is given by

$$I = f_0(0,0) + \frac{1}{\pi}\int d\lambda_1\lambda_2 f_1(\lambda_1,\lambda_2). \tag{60}$$

This result can also be interpreted in terms of the integration measure which is now given by

$$\frac{1}{\pi}d\lambda_1 d\lambda_2 d\sigma d\rho + \frac{1}{\pi}\frac{\delta(\sqrt{\lambda_1^2-\lambda_2^2})}{\sqrt{\lambda_1^2-\lambda_2^2}}d\lambda_1 d\lambda_2. \tag{61}$$

## References


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