9 The Supersymmetric Method of Random Matrix Theory: The One-Point Function

In previous lecture we have seen that the orthogonal polynomials method is a powerful method which has been applied successfully to many problems in Random Matrix Theory. However, this method is not suitable for an important class of problems which have been solved by the supersymmetric method [1, 2]. This method relies on the observation that the average *resolvent* can be written as

$$\langle G(z)\rangle = \frac{1}{N}\left\langle \mathrm{Tr}\frac{1}{z-H}\right\rangle = \frac{1}{N}\frac{\partial}{\partial J}\bigg|_{J=0} Z(J), \tag{1}$$

where the *generating function* is defined by

$$Z(J) = \int DHP(H)\frac{\det(z-H+J)}{\det(z-H)}, \tag{2}$$

and the integral is over the probability distribution of one of the random matrix ensembles defined in the introductory chapter. The determinant can be expressed in terms of Gaussian integrals,

$$\begin{aligned}\frac{\det(z-H+J)}{\det(z-H)} &= \int d\{\psi\}\exp\Big(\sum_{kl}[i\phi_k^*(z-H)_{kl}\phi_l \\ &\qquad + i\chi_k^*(z+J-H)_{kl}\chi_l]\Big)\ ,\end{aligned} \tag{3}$$

where the measure is defined by

$$d\{\psi\} = \prod_{j=1}^{N}\frac{d\phi_j d\phi_j^* d\chi_j d\chi_j^*}{\pi}\ . \tag{4}$$

For convergence the imaginary part of z has to be positive. The integrations over the real and imaginary parts of ϕ_i range over the real axis (the usual commuting, or *bosonic* variables), whereas χ_i and χ_i^* are *Grassmann variables* (i.e., anticommuting, or *fermionic* variables) with integration defined according to the convention that

$$\int d\chi = 0 \quad \text{and} \quad \int \chi d\chi = 1\ . \tag{5}$$

This convention results in the normalization that $Z(0) = 1$.

For simplicity, we consider the GUE $[\beta = 2]$ with probability distribution given by

$$P(H)DH = \mathcal{N}e^{-\frac{N}{2}\mathrm{Tr}H^\dagger H}DH. \tag{6}$$

Also for the supersymmetric method this is mathematically the simplest ensemble. The Gaussian integrals over H can be performed trivially, resulting in the the generating function

$$\begin{aligned} Z(J) &= \int d\{\psi\}\exp\Bigg[-\frac{1}{2N}\mathrm{Trg}\begin{pmatrix} \sum_j \phi_j^*\phi_j & \sum_j \chi_j^*\phi_j \\ \sum_j \chi_j\phi_j^* & \sum_j \chi_j^*\chi_j\end{pmatrix}^2 \\ &\qquad + i\sum_j(\phi_j^* z\phi_j + \chi_j^*(z+J)\chi_j)\Bigg], \end{aligned} \tag{7}$$

where the sums over j run from 1 to N. The quartic terms in ϕ and χ can be expressed as Gaussian integrals by means of a *Hubbard-Stratonovitch transformation.* This results in

$$\begin{aligned} Z(J) &= \int d\{\psi\} d\sigma \exp\Bigg[-\frac{N}{2}\mathrm{Trg}\,\sigma^2 + \\ &\qquad i\sum_j \begin{pmatrix} \phi_j^* \\ \chi_j^* \end{pmatrix} (\sigma+\zeta) \begin{pmatrix} \phi_j \\ \chi_j \end{pmatrix}\Bigg], \end{aligned} \tag{8}$$

where

$$\sigma = \begin{pmatrix} \sigma_{BB} & \sigma_{BF} \\ \sigma_{FB} & i\sigma_{FF} \end{pmatrix} \tag{9}$$

and

$$\zeta = \begin{pmatrix} z & 0 \\ 0 & z+J \end{pmatrix}. \tag{10}$$

The variables σ_{BB} and σ_{FF} are commuting (bosonic) variables that range over the full real axis. Both σ_{BF} and σ_{FB} are Grassmann (fermionic) variables. One easily verifies that by completing the square and carrying out the σ-integrations one recovers the original quartic interaction.

The integrals over the ϕ and the χ variables are now Gaussian and can be performed trivially. This results in the σ-model

$$Z(J) = \int d\sigma \exp\left[-\frac{N}{2}\mathrm{Trg}\sigma^2 + N\mathrm{Trg}\log(\sigma+\zeta)\right]. \tag{11}$$

For $J=0$ we deal with a super-invariant integral, and, according to Wegner's theorem, we have $Z(0)=1$. By shifting the integration variables according to $\sigma \to \sigma - \zeta$ and carrying out the differentiation with respect to J one easily finds that

$$\langle G(z)\rangle = \langle z - i\sigma_{FF}\rangle. \tag{12}$$

In the large N limit, the expectation value of σ_{FF} follows from a *saddle-point* analysis. The saddle point equation for σ_{FF} is given by

$$\sigma_{FF} + iz = 1/\sigma_{FF} \tag{13}$$

resulting in the resolvent

$$\langle G(z)\rangle = \frac{z}{2} - \frac{i}{2}\sqrt{4-z^2}. \tag{14}$$

Using the relation the resolvent and the spectral density we find that the average spectral density is a semi-circle.

However, in this case is is also possible to perform an exact integration instead of a saddle point approximations. The exact result for the partition function is given by

$$Z(J) = \frac{1}{\pi}\int\int dadbd\sigma d\rho \frac{(a+z-\frac{\sigma\rho}{ib+z})^N}{(ib+z)^N} e^{-\frac{N}{2}(a^2+b^2+2\sigma\rho)}. \tag{15}$$

By shifting the integration variables and expanding the Grassmann variables the integral can be rewritten as

$$Z(J) = \frac{1}{\pi}\int d\sigma d\rho \frac{(ib)^N}{a^N}(1-\frac{N\sigma\rho}{aib})(1-N\sigma\rho)e^{-\frac{N}{2}((a-z)^2+(b+iz+iJ)^2)}. \tag{16}$$

After differentiating with respect to J and performing the Grassmann integrations, the resolvent is given by

$$G(z) = \frac{1}{\pi}\int dadb(z-ib)(\frac{(ib)^N}{a^N}+\frac{(ib)^{N-1}}{a^{N+1}})e^{-\frac{N}{2}((a-z)^2+(b+iz)^2)}. \tag{17}$$

The integrals factorize into known integrals. In particular

$$\int_{-\infty}^{\infty} dx(ix)^n e^{-(x+iz)^2} = \frac{\sqrt{\pi}}{2^n}H_n(z). \tag{18}$$

We thus find that the result of the b integral is real. The integral over a is singular. Its calculation was discussed in an article by Guhr [8]. The first observation is that a contains an infinitesimal increment $i\epsilon$. By successive partial integrations we obtain

$$\int da\frac{1}{a^n}e^{-\frac{N}{2}(a-z)^2} = \frac{1}{(n-1)!}\int da\frac{1}{a}\partial_a^{n-1}e^{-\frac{N}{2}(a-z)^2}. \tag{19}$$

In order to find the spectral density we use that

$$\rho(\lambda) = \frac{i}{2\pi}(G(\lambda+i\epsilon)-G(\lambda-i\epsilon)). \tag{20}$$

The discontinuity of $1/a$ is a Dirac delta function. We thus find that

$$\int da\frac{1}{a^n}e^{-\frac{N}{2}(a-z)^2} = \frac{1}{\pi}\frac{1}{(n-1)!}\left(\frac{N}{2}\right)^{(n-1)/2}e^{-\frac{N}{2}z^2}H_{n-1}(z\sqrt{N/2}). \tag{21}$$

Exercise 9.1. Use these results to reproduce the exact result for the spectral density in terms of Hermite polynomials obtained by the orthogonal polynomial method.

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