1 Hirota Equations for 2D Quantum Gravity Models

Lecture and lecture notes by Olindo Corradini

The idea of using Random Matrices to perform calculations in Quantum Gravity was originally due to the need of having non perturbative solutions for the string partition function [1]

$$Z \sim \sum_h \int \mathcal{D}g\mathcal{D}X e^{-S}, \tag{1}$$

at least in some simplified context [2], basically in $D \leq 1$ (space-time dimension). The basic idea relies on a discretization of the string world-sheet in such a way that the functional integral $\int \mathcal{D}g$ over the metric can be calculated as a sum over all random triangulation of the surface (see for example [3]): the surface is represented as an ensemble of contiguous triangles. Random Matrices enter in this context as a powerful generating functional for random triangulation. This goes back to work of 't Hooft [4], which proved that, in a gauge theory with $U(N)$ gauge group, only graphs with planar topology survive in the limit $N \to \infty$, followed by work of Brézin et al who essentially applied the saddle point approximation to evaluate planar graphs.

We consider triangulations of the random surface with equilateral triangles of unit area, so that there is negative (positive) curvature at the vertices i when the number of incident triangles is more (less) than six and zero curvature when $N_i = 6$. [1] The discrete Ricci scalar, at the vertex, is therefore $R_i = 2\pi(6 - N_i)/N_i$. The sum over all such triangulations is thus the discrete analog to the $\int \mathcal{D}g$ over all possible geometries,

$$\sum_h \int \mathcal{D}g \quad \to \sum_{\substack{\text{random} \\ \text{triangulation}}} . \tag{2}$$

It has been shown [5] that each triangulation is dual to a planar diagram which comes from the zero dimensional Matrix Field Theory characterized by the action,

$$S[M] = \frac{1}{2}\mathrm{tr}M^2 + \frac{g}{\sqrt{N}}\mathrm{tr}M^3, \tag{3}$$

where M is a Hermitean matrix. The propagator is given by $\langle M^i_j M^k_l \rangle = \delta^i_l \delta^k_j$ and the three-point vertex results from the M^3 term (see Figure 1). It has to be stressed that the need of "matrix" field theory instead of "ordinary" theory comes from the fact that one needs to have oriented surfaces. In particular, in order to distinguish lower from upper indices hermitian complex matrices are required. [2] The partition function associated with the random triangulation can thus be written as:

[1] $N_i = 6$ is simply a planar hexagon.

[2] For hermitian real matrices $M^i_j = M^j_i$. Therefore there would be no arrows in the propagators and vertices of Figure 1.

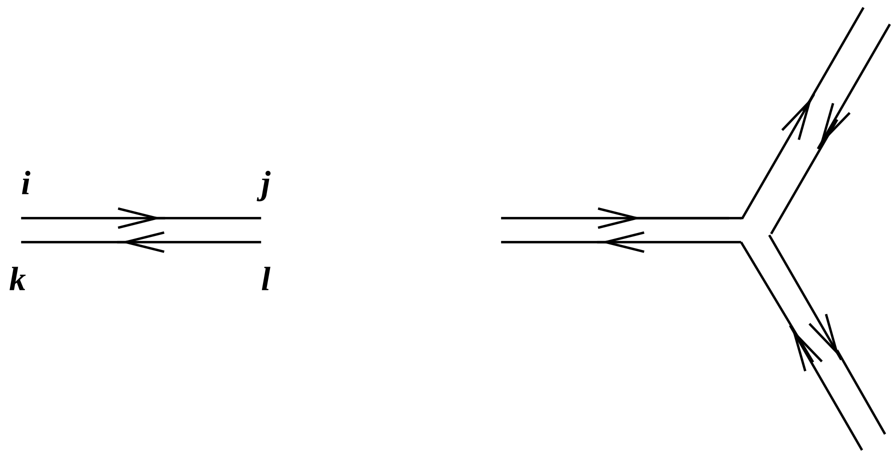

Figure 1.: The hermitian matrix propagator and the hermitian matrix 3-point vertex

$$Z \sim \int dM \, \exp\left(\frac{1}{2}\mathrm{tr}M^2 + \frac{g}{\sqrt{N}}\mathrm{tr}M^3\right). \tag{4}$$

Finally, generalizing to all possible "random polygonulations", one obtains :

$$Z \sim \int dM \, \exp\left(\mathrm{tr}\sum_{n=0}^{\infty} t_n M^n\right), \tag{5}$$

where Z is the partition function of the ensemble of all two-dimensional lattices.

1.1 Hirota equations via orthogonal polynomials method

The partition functions (5) are known to satisfy a hierarchy of integrable differential equations: the so called Kadomtsev-Petviashvili (KP) hierarchy [6]. The global form of this hierarchy is given by the Hirota's bilinear equations [7]. What follow is a derivation of the Hirota equations using the orthogonal polynomials formalism [8].

The partition function can be rewritten as an integral over the eigenvalues of the hermitian (complex) matrices M. Reminding that the Dyson index is $\beta = 2$ we get

$$\begin{aligned} Z_N[t] &= \int \prod_{i=1}^{N} d\lambda_i \exp\left(\sum_{n=1}^{\infty} t_n \lambda_i^n\right) \prod_{i<j}(\lambda_j - \lambda_i)^2 \\ &= \int \prod_{i=1}^{N} d\mu_t(\lambda_i)\, \Delta_N^2(\lambda_1, ..., \lambda_N), \end{aligned} \tag{6}$$

where,

$$d\mu_t(\lambda_i) \equiv d\lambda_i \exp\left(\sum_{n=1}^{\infty} t_n \lambda_i^n\right), \tag{7}$$

and the Vandermonde determinant is, as usual,

$$\Delta_N(\lambda_1, ..., \lambda_N) \equiv \prod_{i<j} (\lambda_j - \lambda_i). \tag{8}$$

The integrations are restricted over the finite real interval $[\lambda_L, \lambda_R]$ such that $d\mu_t(\lambda)$ is integrable for any choice of he coupling constants t_n.

We can introduce the following polynomials,

$$\begin{aligned} P_{N,t}(\lambda) &\equiv \Big\langle \det(\lambda - M) \Big\rangle_{N,t} \\ &= \frac{1}{Z_N} \int \prod_{i=1}^{N} d\mu_t(\lambda_i)\, (\lambda - \lambda_i) \prod_{i<j} (\lambda_j - \lambda_i)^2, \end{aligned} \tag{9}$$

where $\langle \cdots \rangle_{N,t}$ refers to the average with respect to the partition function $Z_N[t]$. The polynomial $P_{N,t}$ is of N^{th} degree in λ because integrations only involve the $\{\lambda_i\}$:

$$P_{N,t}(\lambda) = a_0 + a_1\lambda + \cdots + a_{N-1}\lambda^{N-1} + a_N\lambda_N. \tag{10}$$

We can prove a couple of properties satisfied by these polynomials:

i) P_N's are monic:

$$a_N = \frac{1}{Z_N} \int \prod_{i=1}^{N} d\mu_t(\lambda_i) \prod_{i<j} (\lambda_j - \lambda_i)^2 = 1. \tag{11}$$

ii) P_N's are orthogonal with respect to the measure $d\mu_t(\lambda)$. Let us consider the integral:

$$\begin{aligned} & Z_N \int_{\lambda_L}^{\lambda_R} d\mu_t(\lambda)\, P_{N,t}(\lambda)\lambda^K = Z_N \int_{\lambda_L}^{\lambda_R} d\mu_t(\lambda_{N+1})\, P_{N,t}(\lambda_{N+1})\lambda_{N+1}^K \\ &= \int d\mu_t(\lambda_{N+1}) \int \prod_{i=1}^{N} d\mu_t(\lambda_i) \left[\prod_{i=1}^{N} (\lambda_{N+1} - \lambda_i) \Delta_N(\lambda_1, ..., \lambda_N) \right] \Delta_N(\lambda_1, ..., \lambda_N)\lambda_{N+1}^K \\ &= \int \prod_{i=1}^{N+1} d\mu_t(\lambda_i)\, \Delta_{N+1}(\lambda_1, ..., \lambda_{N+1})\Delta_N(\lambda_1, ..., \lambda_N)\lambda_{N+1}^K. \end{aligned} \tag{12}$$

This expression is horribly asymmetric in the integration variables because only λ_{N+1} appears out of the Vandermonde determinants, but it is possible to make it more symmetric and useful. By renaming the integration variables the previous equation can be rewritten as

$$\int \prod_{i=1}^{N+1} d\mu_t(\lambda_i)\, \Delta_{N+1}(\lambda_1, ..., \lambda_{s-1}, \lambda_{s+1}, ..., \lambda_{N+1}, \lambda_s)\Delta_N(\lambda_1, ..., \lambda_{s-1}, \lambda_{s+1}, ..., \lambda_{N+1})\lambda_s^K \tag{13}$$

and using properties of antisymmetry of Δ_{N+1} we get,

$$(-)^{N+1-s} \int \prod_{i=1}^{N+1} d\mu_t(\lambda_i)\, \Delta_{N+1}(\lambda_1, ..., \lambda_{N+1})\Delta_{N+1}(\lambda_1, ..., \lambda_{s-1}, \hat{\lambda}_s, \lambda_{s+1}, ..., \lambda_{N+1})\lambda_s^K \tag{14}$$

where $\Delta_{N+1}(\lambda_1, ..., \lambda_{s-1}, \hat{\lambda}_s, \lambda_{s+1}, ..., \lambda_{N+1})$ is the minor obtained from $\Delta_{N+1}(\lambda_1, ..., \lambda_{N+1})$ removing the row (λ_s^r: $r = 0, ..., N$) and the column (λ_i^N: $i = 1, ..., N+1$), and is identically equal to $\Delta_N(\lambda_1, ..., \lambda_{s-1}, \lambda_{s+1}, ..., \lambda_{N+1})$; therefore

$$\begin{aligned} & Z_N \int_{\lambda_L}^{\lambda_R} d\mu_t(\lambda)\, P_{N,t}(\lambda)\lambda^K = \\ & = \int \prod_{i=1}^{N+1} d\mu_t(\lambda_i)\, \Delta_{N+1}(\lambda_1, ..., \lambda_{N+1}) \frac{1}{N+1} \\ & \times \left(\sum_{s=0}^{N+1} (-)^{N+1-s} \lambda_s^K \Delta_{N+1}(\lambda_1, ..., \hat{\lambda}_s, ..., \lambda_{N+1}) \right). \end{aligned} \tag{15}$$

Using the Laplace theorem it is easy to recognize the expression between parenthesis as the following determinant:

$$D \equiv \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-1} & \lambda_1^K \\ 1 & \lambda_2 & \cdots & \lambda_2^{N-1} & \lambda_2^K \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_{N+1} & \cdots & \lambda_{N+1}^{N-1} & \lambda_{N+1}^K \end{vmatrix}, \tag{16}$$

which is zero for $K = 0, ..., N-1$, therefore,

$$\int_{\lambda_L}^{\lambda_R} d\mu_t(\lambda)\, P_{N,t}(\lambda)\lambda^K = 0 \qquad\qquad K = 0, ..., N-1, \tag{17}$$

and finally,

$$\int_{\lambda_L}^{\lambda_R} d\mu_t(\lambda)\, P_{N,t}(\lambda) P_{N',t}(\lambda) = 0 \qquad\qquad N' < N. \tag{18}$$

If $K = N$ then $D = \Delta_{N+1}(\lambda_1, ..., \lambda_{N+1})$ and,

$$Z_N \int_{\lambda_L}^{\lambda_R} d\mu_t(\lambda)\, P_{N,t}(\lambda) P_{N,t}(\lambda) = \frac{1}{N+1} Z_{N+1}[t]. \tag{19}$$

We finally summarize,

$$\int_{\lambda_L}^{\lambda_R} d\mu_t(\lambda)\, P_{N,t}(\lambda) P_{N',t}(\lambda) = \delta_{N,N'} \frac{1}{N+1} \frac{Z_{N+1}[t]}{Z_N[t]}. \tag{20}$$

This is the orthogonality relation stated at the beginning.

The relation (20) can be expressed in a even more elegant way, in terms of contour integral,

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} dz \left\langle \det(z - M) \right\rangle_{N,t} \left\langle \frac{1}{\det(z - M)} \right\rangle_{K+1,t} = \delta_{N,K} \tag{21}$$

where the integration contour $\mathcal{C}$ encloses the point $z = 0$ and the interval $[\lambda_L, \lambda_R]$. Indeed the l.h.s. of this equation can be written as

$$\frac{1}{2\pi i}\frac{1}{Z_{K+1}}\int\prod_{j=1}^{K+1}d\mu_t(\lambda_j)\ \Delta^2_{K+1}(\lambda_1,...,\lambda_{N+1})\oint_{\mathcal{C}}dz\prod_{j=1}^{K+1}\frac{1}{z-\lambda_j}P_{N,t}(z). \tag{22}$$

By considering the pole at λ_{K+1} in the previous expression we obtain

$$\begin{aligned}
&\frac{1}{Z_{K+1}}\int d\mu_t(\lambda_{K+1})\ P_{N,t}(\lambda_{K+1})\int\prod_{j=1}^{K}d\mu_t(\lambda_j)\ \frac{1}{\lambda_{K+1}-\lambda_j}\Delta^2_{K+1}(\lambda_1,...,\lambda_{K+1})\\
&=\frac{1}{Z_{K+1}}\int d\mu_t(\lambda_{K+1})\ P_{N,t}(\lambda_{K+1})\int\prod_{j=1}^{K}d\mu_t(\lambda_j)\ (\lambda_{K+1}-\lambda_j)\Delta^2_K(\lambda_1,...,\lambda_K)\\
&=\frac{Z_K[t]}{Z_{K+1}[t]}\int d\mu_t(\lambda)\ P_{N,t}(\lambda)P_{K,t}(\lambda).
\end{aligned} \tag{23}$$

Since each pole gives the same contribution we find

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{\mathcal{C}}dz\Big\langle\det(z-M)\Big\rangle_{N,t}\Big\langle\frac{1}{\det(z-M)}\Big\rangle_{K+1,t}=\\
&=(K+1)\frac{Z_K[t]}{Z_{K+1}[t]}\int d\mu_t(\lambda)\ P_{N,t}(\lambda)P_{K,t}(\lambda)=\delta_{N,K}.
\end{aligned} \tag{24}$$

Notice that $P_{N,t}(\lambda)$ is orthogonal to any polynomial $P_{N',t'}(\lambda)$ of degree less than N where "any" means that the set of coupling constant does not affect this property; therefore:

$$\int d\mu_t(\lambda)\ P_{N,t}(\lambda)P_{N',t'}(\lambda)=0 \qquad\qquad \text{if } N'<N, \tag{25}$$

and for the (24) we have, using the same analysis as before,

$$\begin{aligned}
&\oint_{\mathcal{C}}dz\, e^{\sum_{n=0}^{\infty}(t_n-t'_n)z^n}\Big\langle\det(z-M)\Big\rangle_{N,t}\Big\langle\frac{1}{\det(z-M)}\Big\rangle_{N',t'}=\\
&=N'\frac{Z_{N'-1}[t']}{Z_{N'}[t']}\int d\mu_{t'}(\lambda_{K+1})\ P_{N,t}(\lambda_{K+1})P_{N',t'}(\lambda_{K+1})\exp\left(\sum_{n=0}^{\infty}(t_n-t'_n)\lambda^n_{K+1}\right)\\
&=N'\frac{Z_{N'-1}[t']}{Z_{N'}[t']}\int d\mu_t(\lambda)\ P_{N,t}(\lambda)P_{N',t'}(\lambda)=0.
\end{aligned} \tag{26}$$

The Hirota equations are obtained by expressing the expectation values in terms of the vertex operators,

$$V_\pm(z)\equiv\exp\left(\pm\sum_{n=0}^{\infty}t_nz^n\right)\exp\left(\mp\ln\frac{1}{z}\frac{\partial}{\partial t_0}\mp\sum_{n=1}^{\infty}\frac{z^{-n}}{n}\frac{\partial}{\partial t_n}\right), \tag{27}$$

as

$$V_\pm(z)Z_N[t]=e^{\pm\sum n=0^\infty t_nz^n}\Big\langle\det(z-M)^{\pm1}\Big\rangle_{N,t}Z_N[t], \tag{28}$$

and finally

$$\oint_{\mathcal{C}} dz \left(V_+(z) Z_N[t]\right)\left(V_-(z) Z_{N'}[t']\right) = 0 \qquad N' < N, \tag{29}$$

which is one of the forms of the Hirota equations.

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