Moments of characteristic polynomials enumerate two-rowed lexicographic arrays

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Abstract

A combinatorial interpretation is provided for the moments of characteristic polynomials $Z(U,\theta) = \det(I - e^{-i\theta}U)$ of random unitary matrices. This leads to a rather unexpected consequence of the Keating and Snaith conjecture: the moments of $|\,\xi(1/2 + it)\,|$ turn out to be connected with some increasing subsequences problems (such as the last passage percolation problem).

Conjecture. *(Keating and Snaith)*
The limit:

$$I_m = \lim_{T\to\infty} \frac{1}{(\log T)^{m^2}} \int_0^T |\zeta(1/2+it)|^{2m} dt \quad (1)$$

exists and is equal to a product of two factors, $f(m)$ *and* $a(m)$*, i. e.*

$$I_m = a(m) f(m) \quad (2)$$

The first factor $a(m)$ *is the zeta-function specific part,*

$$a(m) = \prod_p (1 - \frac{1}{p})^{m^2} \sum_{k=0}^{+\infty} \left(\frac{\Gamma(k+m)}{k!\ \Gamma(m)} \right)^2 p^{-k} \quad (3)$$

and the second factor $f(m)$ *is the random matrix (universal) part*

$$f(m) = \lim_{N\to\infty} N^{-m^2} \langle |Z(U,\theta)|^{2m} \rangle_{U(N)} \quad (4)$$

- I provide a combinatorial interpretation for the moments of characteristic polynomials, $\langle |Z(U,\theta)|^{2m} \rangle_{U(N)}$

- These moments are related with two-rowed lexicographic arrays which are generalizations of permutations and words in combinatorics.

- An explicit formula is obtained for the total number of two-rowed lexicographic arrays constructed from the letters of an alphabet of m symbols with the increasing subsequences of the length at most N.

- The random matrix part $f(m)$ defined by equations (1)-(3) is related with the weakly-right/weakly-up lattice model of the last passage percolation.

Increasing subsequences of lexicographic arrays and $\langle |Z(U,\theta)|^{2m} \rangle_{U(N)}$

- Two-rowed array of the size K:
$$A^m_{2,K} = \begin{pmatrix} u_1 & u_2 & \dots & u_K \\ v_1 & v_2 & \dots & v_K \end{pmatrix}$$

- The letters $u_1, u_2, \dots, u_K$ and $v_1, v_2, \dots, v_K$ belong to an alphabet (any ordered set) $\aleph_m$ of m different letters.

- Lexicographic arrays $\Leftrightarrow$
$$\begin{cases} u_j < u_{j+1} \\ \text{if } u_j = u_{j+1} \text{ then } v_j < v_{j+1}. \end{cases}$$

- Lexicographic array $\leftrightarrow$ pair of *semi-standard* Young tableaux

- $l(A^m_{2,K})$ -length of the maximal (weakly) increasing subsequence of $A^m_{2,K}$

- $R^K_{m,N} = \sharp$ of $A^m_{2,K}$ with $l(A^m_{2,K}) \leq N$
  $R_{m,N} = \sum_{K=0}^{+\infty} R^K_{m,N}$

- $d_\lambda(m)$ - $\sharp$ of semi-standard Young tableaux of shape λ with entries from $[1, 2, \ldots, m]$

The Robinson-Schensted-Knuth correspondence gives:

$$R_{m,N} = \sum_{K=0}^{+\infty} \sum_{\lambda \vdash K,\ \lambda_1 \leq N} d^2_\lambda(m) = \langle |Z(U,\theta)|^{2m} \rangle_{U(N)}$$

Combinatorial interpretation: $\langle |Z(U,\theta)|^{2m}\rangle_{U(N)}$ **equal to the number of two-rowed lexicographic arrays constructed from an alphabet of** m **symbols with the weakly increasing subsequences of the** N **elements at most.**

- The Keating and Snaith formula:

$$\langle |Z(U,\theta)|^{2m}\rangle_{U(N)} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2m)}{[\Gamma(j+m)]^2} \tag{5}$$

- The equality between the number of arrays $R_{m,N}$ and the moments of characteristic polynomials gives:

$$R_{m,N} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2m)}{[\Gamma(j+m)]^2} \tag{6}$$

Example. *Alphabet:* $\aleph_m = (a, b)$, $\quad m = 2$.
A typical lexicographic array:

$$\begin{pmatrix} a & a & b & b \\ a & b & b & b \end{pmatrix}$$

An example of a two-rowed non-lexicographic array :

$$\begin{pmatrix} a & a & b & b \\ a & b & a & b \end{pmatrix}$$

(second letter of the word $abab$, b, *is larger than the third letter of this word* a*)*

All possible two-rowed lexicographic arrays from $\aleph_m$ *with the weakly increasing subsequences of the length at most* $N = 2$*:*

$$\binom{\emptyset}{\emptyset} \quad \binom{a}{a} \quad \binom{a}{b} \quad \binom{b}{b} \quad \binom{b}{a} \quad \binom{ab}{aa}$$

$$\binom{aa}{ab} \quad \binom{ab}{ba} \quad \binom{aa}{bb} \quad \binom{aa}{aa} \quad \binom{bb}{ab} \quad \binom{bb}{bb} \quad \binom{ab}{bb} \quad \binom{bb}{aa}$$

$$\binom{ab}{ab} \quad \binom{abb}{bab} \quad \binom{aab}{bba} \quad \binom{abb}{baa} \quad \binom{aab}{aba} \quad \binom{aabb}{bbaa}$$

There are 20 arrays with the required properties. The formula (6) obtained above gives precisely this number, i.e.

$$\mathsf{R}_{m,N} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2m)}{[\Gamma(j+m)]^2} = 20 \qquad (m = 2, N = 2)$$

- $X(i,j), \;\; i,j \in [1,2,3,\cdots,m]$:
  a planar array of independent integers

$$X(i,j) = \begin{matrix} * & * & * & *\text{ij} & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{matrix}$$

  *ij $\equiv$ the passage time at (ij)

- $(1,1) \searrow (m,m)$ $\equiv$ the set of down/right paths from (1,1) to (m,m)

- $T(m,m) = \max\limits_{\pi \in (1,1) \searrow (m,m)} \sum\limits_{i,j \in \pi} *\text{ij} \equiv$ last passage percolation time to travel from (1,1) to (m,m)

- $\sharp$ of $X(i,j)$ with $T(m,m) \leq n = \langle |Z(U,\theta)|^{2m} \rangle_{U(N)}$
  $= \prod\limits_{j=1}^{n} \frac{\Gamma(j)\Gamma(j+2m)}{[\Gamma(j+m)]^2}$