Moments of characteristic polynomials enumerate two-rowed lexicographic arrays

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Abstract

A combinatorial interpretation is provided for the moments of characteristic polynomials $Z(U, \theta) = \det(I - e^{-i\theta}U)$ of random unitary matrices. This leads to a rather unexpected consequence of the Keating and Snaith conjecture: the moments of $|\xi(1/2 + it)|$ turn out to be connected with some increasing subsequences problems (such as the last passage percolation problem).
Conjecture. (Keating and Snaith)
The limit:

\[ I_m = \lim_{T \to \infty} \frac{1}{(\log T)^m} \int_0^T |\zeta(1/2 + it)|^{2m} dt \quad (1) \]

exists and is equal to a product of two factors, \( f(m) \) and \( a(m) \), i.e.

\[ I_m = a(m) f(m) \quad (2) \]

The first factor \( a(m) \) is the zeta-function specific part,

\[ a(m) = \prod_p (1 - \frac{1}{p})^{m^2} \sum_{k=0}^{+\infty} \left( \frac{\Gamma(k + m)}{k! \Gamma(m)} \right)^2 p^{-k} \quad (3) \]

and the second factor \( f(m) \) is the random matrix (universal) part

\[ f(m) = \lim_{N \to \infty} N^{-m^2} \langle |Z(U, \theta)|^{2m} \rangle_{U(N)} \quad (4) \]
• I provide a combinatorial interpretation for the moments of characteristic polynomials, \( \langle |Z(U, \theta)|^{2m} \rangle_{U(N)} \)

• These moments are related with two-rowed lexicographic arrays which are generalizations of permutations and words in combinatorics.

• An explicit formula is obtained for the total number of two-rowed lexicographic arrays constructed from the letters of an alphabet of \( m \) symbols with the increasing subsequences of the length at most \( N \).

• The random matrix part \( f(m) \) defined by equations (1)-(3) is related with the weakly-right/weakly-up lattice model of the last passage percolation.
Increasing subsequences of lexicographic arrays and $\langle |Z(U, \theta)|^{2m} \rangle_{U(N)}$

- Two-rowed array of the size $K$:
  $$A_{2,K}^m = \begin{pmatrix} u_1 & u_2 & \ldots & u_K \\ v_1 & v_2 & \ldots & v_K \end{pmatrix}$$

- The letters $u_1, u_2, \ldots, u_K$ and $v_1, v_2, \ldots, v_K$ belong to an alphabet (any ordered set) $\mathbb{N}_m$ of $m$ different letters.

- Lexicographic arrays $\Leftrightarrow$
  $$\begin{cases} u_j < u_{j+1} \\ \text{if } u_j = u_{j+1} \text{ then } v_j < v_{j+1}. \end{cases}$$

- Lexicographic array $\leftrightarrow$ pair of semi-standard Young tableaux
• \( l(A_{2,K}^m) \) - length of the maximal (weakly) increasing subsequence of \( A_{2,K}^m \)

• \( R_{m,N}^K = \# \) of \( A_{2,K}^m \) with \( l(A_{2,K}^m) \leq N \)

\[
R_{m,N} = \sum_{K=0}^{+\infty} R_{m,N}^K
\]

• \( d_\lambda(m) \) - \# of semi-standard Young tableaux of shape \( \lambda \) with entries from \([1,2,\ldots,m]\)

The Robinson-Schensted-Knuth correspondence gives:

\[
R_{m,N} = \sum_{K=0}^{+\infty} \sum_{\lambda \vdash K, \lambda_1 \leq N} d_\lambda^2(m) = \langle |Z(U, \theta)|^{2m} \rangle_{U(N)}
\]
Combinatorial interpretation: \( \langle |Z(U, \theta)|^{2m} \rangle_{U(N)} \) equal to the number of two-rowed lexicographic arrays constructed from an alphabet of \( m \) symbols with the weakly increasing subsequences of the \( N \) elements at most.

- The Keating and Snaith formula:

\[
\langle |Z(U, \theta)|^{2m} \rangle_{U(N)} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j + 2m)}{[\Gamma(j + m)]^2}
\]

(5)

- The equality between the number of arrays \( R_{m,N} \) and the moments of characteristic polynomials gives:

\[
R_{m,N} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j + 2m)}{[\Gamma(j + m)]^2}
\]

(6)
**Example.** Alphabet: $\mathbb{A}_m = (a, b), \quad m = 2$.
A typical lexicographic array:

\[
\begin{pmatrix}
a & a & b & b \\
 a & b & b & b \\
\end{pmatrix}
\]

An example of a two-rowed non-lexicographic array:

\[
\begin{pmatrix}
a & a & b & b \\
 a & b & a & b \\
\end{pmatrix}
\]

(second letter of the word $abab$, $b$, is larger than the third letter of this word $a$)

All possible two-rowed lexicographic arrays from $\mathbb{A}_m$ with the weakly increasing subsequences of the length at most $N = 2$:

\[
\begin{aligned}
(\emptyset) & \quad (a) & \quad (a) & \quad (b) & \quad (b) & \quad (b) & \quad (ab) \\
(\emptyset) & \quad (a) & \quad (a) & \quad (b) & \quad (b) & \quad (a) & \quad (ab) \\
(aa) & \quad (ab) & \quad (aa) & \quad (aa) & \quad (bb) & \quad (bb) & \quad (ab) & \quad (bb) \\
(ab) & \quad (ba) & \quad (bb) & \quad (aa) & \quad (ab) & \quad (bb) & \quad (bb) & \quad (aa) \\
(ab) & \quad (bb) & \quad (aab) & \quad (aab) & \quad (abb) & \quad (abb) & \quad (aabb) & \quad (bbaa) \\
(ab) & \quad (bab) & \quad (bbba) & \quad (bbaa) & \quad (aba) & \quad (bbba) & \quad (bbaa) & \quad (bab)
\end{aligned}
\]

There are 20 arrays with the required properties. The formula (6) obtained above gives precisely this number, i.e.

\[
R_{m,N} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j + 2m)}{[\Gamma(j + m)]^2} = 20 \quad (m = 2, N = 2)
\]
• $X(i, j), \ i, j \in [1, 2, 3, \ldots, m]$:
  a planar array of independent integers

  
  $$X(i, j) = \begin{array}{cccc}
  * & * & * & *ij * \\
  * & * & * & * * * * \\
  * & * & * & * * * * \\
  * & * & * & * * * * \\
  * & * & * & * * * * \\
  \end{array}$$

  *ij \equiv \text{the passage time at (ij)}$

• $(1, 1) \searrow (m, m) \equiv \text{the set of down/right paths from (1,1) to (m,m)}$

• $T(m, m) = \max_{\pi \in (1,1) \searrow (m,m)} \sum_{i,j \in \pi} *ij \equiv \text{last passage percolation time to travel from (1,1) to (m,m)}$

• # of $X(i, j)$ with $T(m, m) \leq n = \langle |Z(U, \theta)|^{2m} \rangle_{U(N)}$
  $$= \prod_{j=1}^{n} \frac{\Gamma(j)\Gamma(j+2m)}{[\Gamma(j+m)]^2}$$