Eq. (2.20) establishes the temperature dependence of the total radiation intensity: (see eq (2.10))

(2.21) $$S(T) = \frac{c}{4}\, u(T) \equiv \sigma_{SB}\, T^4$$

Here, we have defined the Stefan-Boltzmann ~~law~~ constant

(2.22) $$\sigma_{SB} = 5.67 \cdot 10^{-5} \frac{\text{erg}}{\text{cm}^2\,\text{s}\,\text{K}^4}$$

$$= 5{,}67 \cdot 10^{-8} \frac{\text{W}}{\text{m}^2\,\text{K}^4}$$

## II.4 Wien's displacement law

We have seen from the previous section that applying classical thermodynamic reasoning to the blackbody radiation, one can derive the functional dependence of $u(T)$.

Wien considered blackbody radiation under adiabatic compression. He found

(2.23) $$u(f,T)\,df = \cancel{f^3 g\left(\tfrac{f}{T}\right) df}$$ Wien's law (1893)

$$= f\; g\left(\frac{f}{T}\right) df \qquad g = \text{arbitrary function}$$

Since $f = \frac{c}{\lambda}$, it follows immediately that the spectral energy density distribution has a maximum at a wavelength $\lambda_{max}$ inversely proportional to $T$

(2.24) $$\lambda_{max} \cdot T = \text{const} = 0{,}29 \text{ cm} \cdot \text{K}$$

Wien's displacement law

## II.5 Rayleigh-Jeans Formula; Ultraviolett Catastrophe

From classical electrodynamics, we know

Kirchhoff: $u(f,t)$ is universal function

Stefan-Boltzmann: $S(T) = \sigma_{SB} T^4$

Wien: a.) $u(f,T) = f^3 g(\frac{f}{T})$

b.) $\lambda_{max} \cdot T = const.$

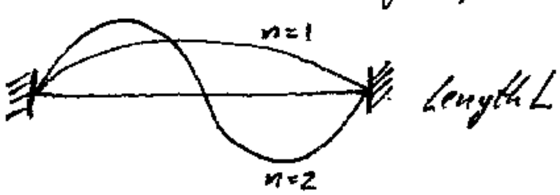
Now, we want to know the spectral distribution of $u$ (i.e. $f$-dependence)

To this end: A: calculate the number of modes between $f$ and $f+df$

B: to obtain $u(f,T)$, multiply by average energy per mode and divide by volume

A. How many modes fit into a cavity?

Waves have vanishing amplitudes at wall if they are static




(2.25) $$L = n \cdot \frac{\lambda}{2} \quad , n = 1, 2, 3, \ldots \quad \text{integer}$$

(2.26) $$\equiv n \cdot \frac{\pi}{k}$$

(2.27) $$k \equiv \frac{2\pi}{\lambda}$$ is called 'wave vector'

The n-th wave vector $k_n = n\frac{\pi}{L}$ characterizes the n-th mode

(2.28) $$\delta k = k_{n+1} - k_n = \frac{\pi}{L}$$ separation between modes in k-space

Typically, one allows for the range $-\infty < k < \infty$ rather than $0 < k < \infty$ (amounts to multiplication by factor $1/2$).
The number of modes in interval $dk$ is then

(2.29) $$dN = \frac{1}{2}\,\frac{dk}{\delta k} = \frac{L}{2\pi}\,dk$$

In 3 dimensions, the same argument leads to

(2.30) $$dN(\vec{k}) = dN_x\,dN_y\,dN_z = \underbrace{L_x L_y L_z}_{\text{Volume}}\,\frac{1}{(2\pi)^3}\,dk_x\,dk_y\,dk_z$$

$$= V\,\frac{d^3k}{(2\pi)^3}$$

The 'volume' in k-space is

(2.31) $$d^3k = 4\pi\,k^2\,dk$$


In the vacuum

(2.32) $$|\vec{k}| = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}$$

Inserting into (2.30)

(2.33) $$dN(\vec{k}) = V\,\frac{4\pi k^2 dk}{(2\pi)^3} = V\,\frac{4\pi}{(2\pi)^3}\left(\frac{2\pi f}{c}\right)^2 \frac{2\pi}{c}df = V\,\frac{4\pi}{c^3}\,f^2 df$$

Each el.mag wave comes with 2 polarizations. Thus

(2.34) $$dn(f) = \frac{8\pi}{c^3}\,f^2 df$$ density of modes between $f$ and $f+df$

## B. What is the average energy per wave mode?

Rayleigh-Jeans start from classical thermodynamics:

If a mode has energy $E$ above the lowest lying energy $E_0$, then it is found in a system of ~~one~~ temperature $T$ with the probability

(2.35) $$P(E) = \mathcal{N}\, e^{-(E-E_0)/k_B T} \quad , \quad \mathcal{N} = \text{normalization} = P(E_0)$$

where the Boltzmann constant is

(2.36) $$k_B = 1.38 \cdot 10^{-23}\, J/K = 8.62 \cdot 10^{-5}\, eV/K$$

The average energy $\bar{E}_{RJ}$ of a mode in the blackbody spectrum is then

(2.37) $$\bar{E}_{RJ} = \frac{\int_0^\infty dE\, E\, P(E)}{\int_0^\infty dE\, P(E)} = \frac{\int_0^\infty dE\, E\, e^{-E/k_B T}}{\int_0^\infty dE\, e^{-E/k_B T}} = \underline{\underline{k_B T}}$$

The spectral energy density $u(f,T)$ of blackbody radiation is then

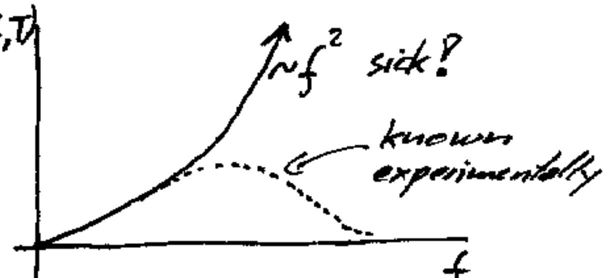
$$u_{RJ}(f,T) = \bar{E}_{RJ}\, dn(f)$$

(2.38) $$= \underline{\underline{\frac{8\pi}{c^3} f^2 k_B T\, df}}$$

This has a sick ultra violett behavior (UV catastrophe)

(2.39) $$u_{RJ}(T) = \int_0^\infty u_{RJ}(f,T)\, df$$
$$= \text{infinity}$$

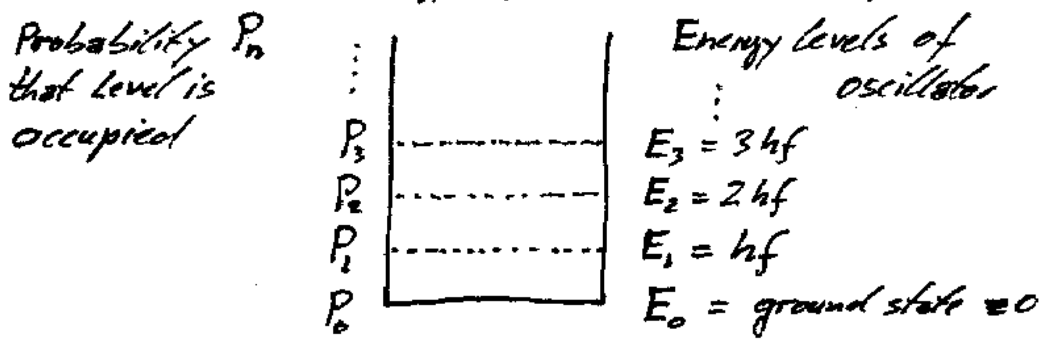
cloud over classical physics



## II.6 Planck's solution of the UV catastrophe

Properties of wall of the blackbody cavity can be chosen arbitrarily, since $u(f,T)$ is universal.
Planck views the wall as a collection of oscillators, which are in thermodynamic equilibrium with elmag. field in cavity.

Planck assumes that for each frequency $f$ in the cavity, the oscillators can emit or absorb only integer multiples of an elementary energy unit $hf$.
That means, the energy of the oscillator is quantized




We write again the Boltzmann distribution, which is based on classical statistical mechanics (2.35)

(2.40)
$$P_n = P_0 \exp\left[-\frac{E_n}{k_B T}\right] \quad ; \quad E_n = nhf$$
$$= P_0 \, x^n$$

where we introduced the short hand

(2.41)
$$x = e^{-\frac{hf}{k_B T}}$$

The average energy in the mode of frequency $f$ is now [compare with (2.37)]

(2.42)
$$\bar{E}_{Planck} = \frac{\sum_n E_n P_n}{\sum_n P_n} = hf \, \frac{\sum_n n x^n}{\sum_n x^n}$$

To calculate (2.42), determine first the norm

$$S(x) \equiv \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$$

$$\Rightarrow \quad -x\,S(x) = -x - x^2 - x^3 - \ldots$$

$$\Rightarrow \quad S(x) - x\,S(x) = 1$$

$$= S(x)\,(1-x)$$

$$(2.43) \quad \Rightarrow \quad \underline{\underline{S(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}}$$

Now comes the numerator of (2.42)

$$(2.44) \quad \sum_{n=0}^{\infty} n x^n = x \sum_{n=0}^{\infty} n x^{n-1} = x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = x \frac{d}{dx} S(x)$$

$$= x \frac{d}{dx} \frac{1}{1-x}$$

$$= x \frac{+1}{(1-x)^2}$$

Inserting (2.43) and (2.44) into (2.42), we find

$$(2.45) \quad \overline{E}_{Planck} = hf \frac{\sum_n n x^n}{\sum_n x^n} = hf \frac{\frac{x}{(1-x)^2}}{\frac{1}{1-x}} = hf \frac{x}{1-x}$$

$$= hf \frac{1}{1/x - 1}$$

$$= \underline{\underline{\frac{hf}{e^{hf/k_B T} - 1}}}$$

$h$ = Planck's constant

$$(2.46) \quad \hbar \equiv \frac{h}{2\pi} = 1.054 \cdot 10^{-27}\,\text{erg}\cdot\text{s} = 1.054 \cdot 10^{-34}\,\text{J}\,\text{s}$$