The magnetic quantum number

The solution of \( \frac{\partial^2 \phi}{\partial \psi^2} + \kappa^2 \phi = 0 \) in (5.24) is

(5.26)\hspace{1cm} 1) \text{case } \kappa = 0 \quad \phi(\psi) = \alpha + \beta \psi

(5.27)\hspace{1cm} 2) \text{case } \kappa \neq 0 \quad \phi(\psi) = \alpha e^{i\kappa \psi} + \beta e^{-i\kappa \psi}

According to postulate 1, \( \phi(\psi) \) must be unique, thus

(5.28)\hspace{1cm} \phi(\psi) = \phi(\psi + 2\pi)

This leads to the conditions:

\text{case 1: } \beta = 0 \quad \Rightarrow \quad \phi = \text{const} \left. e^{im\psi} \right|_{m=0, \pm 1, \pm 2, \pm 3, \ldots}

\text{case 2: } e^{i\kappa 2\pi} = e^{-i\kappa 2\pi} = 1 \quad \Rightarrow \quad \kappa \text{ is an integer}

\Rightarrow \quad \phi(\psi) = \text{const} \left. e^{im\psi} \right|_{m=0, \pm 1, \pm 2, \pm 3, \ldots}

Normalization: \( \int_0^{2\pi} \phi(\psi) \phi(\psi) d\psi = \text{const}^2 \int_0^{2\pi} e^{im\psi} e^{-im\psi} d\psi \)

(5.29)\hspace{1cm} \Rightarrow \quad \phi(\psi) = \left. \frac{1}{\sqrt{2\pi}} e^{im\psi} \right|_{m=0, \pm 1, \pm 2, \pm 3, \ldots}

(5.30)\hspace{1cm} m = \text{magnetic quantum number}
4. The orbital quantum number

Knowing that \( x^2 = m^2 \), \( m \) integer, we write (5.23) as (divide by \( \sin^2 \theta \))

\[
\frac{\partial^2}{\partial y^2} \theta + \cot y \frac{\partial}{\partial y} \theta - \frac{m^2}{\sin^2 \theta} \theta = -2\theta
\]

(5.31)

It can be shown (mathematical problem, not explained here) that physically reasonable solutions exist only for

\[
\lambda = \ell(\ell+1), \quad \ell = 0, 1, 2, 3, \ldots \text{ integer}
\]

(5.32)

\[
|\ell| \leq \ell
\]

(5.33)

The integer \( \ell \) in (5.32) is called the orbital quantum number \( \ell \).

The solutions of (5.31) are known as "associated Legendre polynomials" \( P_{\ell}^{\ell-1}(x = \cos \theta) \), defined as

\[
P_{\ell}^{\ell-1}(x) = \left( 1-x^2 \right)^{\frac{1-\ell}{2}} \frac{d^{\ell+1}}{dx^{\ell+1}} (x^2-1)^{\ell}
\]

(5.34)

So, the solution of the angular equation (5.21) is

\[
Y_{\ell, m}(x, \theta, \phi) = \alpha_{\ell m} \, P_{\ell}^{m}(\cos \theta) e^{im\phi}
\]

(5.35)

\[= \Theta(x) \cdot \phi(\phi)\]
The solutions $Y_{l,m}(\theta, \phi)$ are called spherical harmonics.

The normalization in (5.35) reads

$$x_{l,m} = \sqrt{\frac{2l+1}{4\pi} \frac{(l+m)!}{(l-m)!}}$$

Examples of spherical harmonics:

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$
$$Y_{1,1} = -\frac{\sqrt{3}}{\sqrt{8\pi}} \sin \theta e^{i\phi}$$
$$Y_{1,0} = \frac{\sqrt{3}}{\sqrt{4\pi}} \cos \theta$$
$$Y_{1,-1} = \frac{\sqrt{3}}{\sqrt{8\pi}} \sin \theta e^{-i\phi}$$

So, for orbital quantum number $l=0$, we have $m=0 \Rightarrow 1$ spherical harmonic

$l=1$
$$m=0, \pm 1 \Rightarrow 3 \text{ spherical harmonics}$$
$$m=0, \pm 1, \pm 2 \Rightarrow (2l+1)$$

Homework: check $Y_{l,m}$ given in Table 8.3 of Serway by using (5.34), (5.36)
V.11 Quantization of angular momentum

In homework 8, you'll derive expressions for the squared angular momentum operator \( \hat{L}^2 \) and its components \( \hat{L}_x, \hat{L}_y, \hat{L}_z \) (see eqns. (4.78)-(4.75)) in terms of spherical coordinates. In particular

\[
\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2
\]

\[
= -\hbar^2 \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)
\]

\[
\equiv \hbar^2 \hat{Q}_2
\]

Here, \( \hat{Q}_2 \) is the so-called Legendre operator. It is exactly the operator which appears on the r.h.s. of (5.9) and which defines the angular equation.

The spherical harmonics \( Y_{\ell m}(\theta, \phi) \) are eigenfunctions of the Legendre operator with eigenvalues \( \lambda = \ell (\ell + 1) \)

\[
\hat{Q}_2 Y_{\ell m}(\theta, \phi) = \ell (\ell + 1) Y_{\ell m}(\theta, \phi)
\]

\( \hat{L}^2 \) in (5.11) corresponds to a measurable quantity, the total angular momentum squared. Eq. (5.13) implies that the angular momentum is quantized, \( \hat{L}^2 \to \ell (\ell + 1) \hbar^2 \)

\[
|\hat{L}| = \sqrt{\hat{L}^2} = \sqrt{\ell (\ell + 1)} \hbar
\]

(recall postulate 3)
In homework 8, we derive for the $z$-component of the angular momentum operator

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}$$

(5.42)

Acting with $\hat{L}_z$ on the spherical harmonics (5.35), we find

$$\hat{L}_z \ Y_l^m(\theta, \phi) = \hbar m \ Y_l^m(\theta, \phi)$$

(5.43)

The spherical harmonics $Y_l^m$ are eigenfunctions of the $z$-component of the angular momentum $\hat{L}_z$ with eigenvalue $m$.

Since $\hat{L}_x, \hat{L}_y, \hat{L}_z$ commute with $\hat{L}^2$ [see (4.79)] but not amongst themselves [see (4.76), (4.77), (4.78)], this means that $\hat{L}^2$ and $\hat{L}_z$ can be measured precisely simultaneously. That's why they have the same eigenfunctions $Y_l^m$.

Selecting the $z$-direction, note that

$$\cos \vartheta = \frac{L_z}{|L|} = \frac{m}{\sqrt{l(l+1)}}$$

(5.44)

The orientation of the angular momentum is quantized (so-called space-quantization)

(5.45) total length of vector: $\sqrt{l(l+1)} \cdot \hbar$