V.3 The magnetic quantum number

The solution of $\frac{\partial^2 \phi}{\partial \varphi^2} + \varkappa^2 \phi = 0$ in (5.24) is

(5.26) 1.) case $\varkappa^2 = 0$ $\qquad \phi(\varphi) = a + \beta\varphi$

(5.27) 2.) case $\varkappa^2 \neq 0$ $\qquad \phi(\varphi) = a\, e^{i\varkappa\varphi} + b\, e^{-i\varkappa\varphi}$

According to postulate 1, $\phi(\varphi)$ must be unique, thus

(5.28)
$$\phi(\varphi) = \phi(\varphi + 2\pi)$$

This leads to the conditions

case 1: $\beta = 0 \Rightarrow \phi = \text{const.}$

case 2: $e^{i\varkappa 2\pi} = e^{-i\varkappa 2\pi} = 1 \Rightarrow \varkappa$ is an integer

$$\Rightarrow \phi(\varphi) = \text{const.}\, e^{im\varphi} \quad , \; m = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Normalization: $\int_0^{2\pi} \phi^*(\varphi)\, \phi(\varphi)\, d\varphi = \text{const}^2 \int_0^{2\pi} e^{-im\varphi}\, e^{im\varphi}\, d\varphi$

(5.29)
$$\Rightarrow \boxed{\phi(\varphi) = \frac{1}{\sqrt{2\pi}}\, e^{im\varphi}}$$

(5.30)
$$\boxed{m = 0, \pm 1, \pm 2, \pm 3, \ldots}$$

m = magnetic quantum number

V.4 The orbital quantum number

Knowing that $\varkappa^2 = m^2$, m integer, we write (5.23) as (divide (5.23) by $\sin^2\vartheta$)

$$\frac{\partial^2}{\partial\vartheta^2}\Theta + \cot\vartheta \frac{\partial}{\partial\vartheta}\Theta - \frac{m^2}{\sin^2\vartheta}\Theta = -\lambda\Theta \tag{5.31}$$

It can be shown (mathematical problem, not explained here) that physically reasonable solutions exist only for

$$\lambda = \ell(\ell+1) \quad , \quad \ell = 0,1,2,3,\ldots \text{ integer} \tag{5.32}$$

$$|m| \leq \ell \tag{5.33}$$

The integer ℓ in (5.32) is called the <u>orbital quantum number</u> ℓ.

The solutions of (5.31) are known as "associated <u>Legendre polynomials</u>" $P_\ell^{|m|}(x=\cos\vartheta)$, defined as

$$P_\ell^{|m|}(x) = \frac{(1-x^2)^{\frac{|m|}{2}}}{2^\ell\,\ell!}\,\frac{d^{\ell+|m|}}{dx^{\ell+|m|}}(x^2-1)^\ell \tag{5.34}$$

So, the solution of the angular equation (5.21) is

$$Y_{\ell,m}(\vartheta,\varphi) = \alpha_{\ell m}\,P_{\ell m}(\cos\vartheta)\,e^{im\varphi} \tag{5.35}$$

$$= \Theta(\vartheta)\cdot\Phi(\varphi)$$

The solutions $Y_{\ell,m}(\vartheta,\varphi)$ are called <u>spherical harmonics</u>

The normalization in (5.35) reads

(5.36) $$\alpha_{\ell,m} = \sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}$$

Examples of spherical harmonics:

(5.37)
$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$
$$Y_{1,1} = -\sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{i\varphi}$$
$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\vartheta$$
$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{-i\varphi}$$

So, for orbital quantum number

$\ell = 0$, we have $m=0 \Rightarrow 1$ spherical harmonic
$\ell = 1$ $\qquad m=0,\pm1 \Rightarrow 3$ spherical harmonics
ℓ $\qquad m=0,\pm1,..,\pm\ell \Rightarrow (2\ell+1)$ "

Homework: check $Y_{\ell,m}$ given in Table 8.3 of Serway

by using (5.34), (5.36)

V.5 Quantization of angular momentum

In homework 8, you'll derive expressions for the squared angular momentum operator $\hat{L}^2$ and its components $\hat{L}_x, \hat{L}_y, \hat{L}_z$ (see eqns. (4.73) – (4.75)) in terms of spherical coordinates. In particular

(5.38) $$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$= -\hbar^2 \left(\frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial\varphi^2} \right)$$

(5.39) $$\equiv \hbar^2 \hat{O}_L$$

Here, $\hat{O}_L$ is the so-called <u>Legendre-operator</u>. It is exactly the operator which appears on the r.h.s. of (5.19) and which defines the angular equation.

The spherical harmonics $Y_{\ell,m}(\vartheta,\varphi)$ are eigenfunctions of the Legendre operator with eigenvalues $\lambda = \ell(\ell+1)$

(5.40) $$\hat{O}_L \, Y_{\ell,m}(\vartheta,\varphi) = \ell(\ell+1)\, Y_{\ell,m}(\vartheta,\varphi)$$

$\hat{L}^2$ in (5.11) corresponds to a measurable quantity, the total angular momentum squared. Eq. (5.13) implies that the <u>total angular momentum</u> is quantized, $L^2 \to \ell(\ell+1)\hbar^2$

(5.41) $$|\vec{L}| = \sqrt{L^2} = \sqrt{\ell(\ell+1)}\,\hbar$$

(recall postulate 3)

In homework 8, we derive for the z-component of the angular momentum operator

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi} \tag{5.42}$$

Acting with $\hat{L}_z$ on the spherical harmonics (5.35), we find

$$\hat{L}_z\, Y_{\ell,m}(\vartheta,\varphi) = \hbar m\, Y_{\ell,m}(\vartheta,\varphi) \tag{5.43}$$

The spherical harmonics $Y_{\ell,m}$ are eigenfunctions of the z-component of the angular momentum $\hat{L}_z$ with eigenvalue m.

Since $\hat{L}_x, \hat{L}_y, \hat{L}_z$ commute with $\hat{L}^2$ [see (4.79)] but not amongst themselves [see (4.76), (4.77), (4.78)], this means that $\hat{L}^2$ and $\hat{L}_z$ can be measured <u>precisely</u> simultaneously. That's why they have the same eigenfunctions $Y_{\ell,m}$

Selecting the z-direction, note that

$$\cos\vartheta = \frac{L_z}{|\vec{L}|} = \frac{m}{\sqrt{\ell(\ell+1)}} \tag{5.44}$$

The orientation of the angular momentum is quantized (so-called <u>space-quantization</u>)

(5.45) total length of vector: $\sqrt{\ell(\ell+1)}\cdot\hbar$