PHY251 Modern Physics, Homework 8
29 Mar - 5 Apr Lecturer: Urs Achim Wiedemann
e-mail: wiedemann@tonic.physics.sunysb.edu Tel: 631-632-4489 Fax: 631-632-9718

1. Consider an infinitely extended potential barrier in one dimension

$$\begin{aligned} V(x) &= 0 \qquad \text{for} \quad x < 0\,, \\ V(x) &= V \qquad \text{for} \quad x > 0\,. \end{aligned}$$

Consider a particle with energy $E < V$, described by the wavefunction

$$\begin{aligned} \psi(x) &= \frac{1}{2}(1+i)\,e^{ikx} + \frac{1}{2}(1-i)\,e^{-ikx} \qquad \text{for}\, x \le 0 \\ \psi(x) &= e^{-kx} \qquad \text{for}\, x \ge 0\,. \end{aligned}$$

i) Show by explicit calculation, that this is a solution of the time-independen Schrödinger equation with reflection coefficient $R = 1$ for specific values of k. How is k related to the energy E for $x < 0$ and $x > 0$ for this to be a solution? (Hint: if you don't know what to do, search in Serway for parts of this problem.)

2. A particle of mass M moves in a three-dimensional box with edge lengths L_1, L_2 and L_3. Find the energies of the six lowest states if $L_1 = L$, $L_2 = 2L$ and $L_3 = 2L$. Which of these energies are degenerate?

3. The normalized ground-state wavefunction for the electron in the hydrogen atom is

$$\psi(r,\theta,\varphi) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}\,,$$

where r is the radial coordinate of the electron and a_0 is the Bohr radius. i) Sketch the wavefunction versus r. ii) Show that the probability of finding the electron between r and $r + dr$ is given by $|\psi(r)|^2 4\,\pi\,r^2$. iii) Sketch the probability versus r and from your sketch find the radius at which the electron is most likely to be found. iv) Show that the wavefunction as given is normalized. v) Find the probability of locating the electron between $r_1 = a_0/2$ and $r_2 = 3a_0/2$.

4. i) Calculate the commutator between the components of the angular momentum $\left[\hat{L}_x, \hat{L}_y\right]$. Use cartesian coordinates.
ii) Express $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$ in spherical coordinates. Do a proper calculation and check your result against the explicit expressions given in Serway. Do not give up before you succeed to reproduce them!
iii) Show by explicit calculation that

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

Did you encounter this operator already?

5. In two dimensions, the relation between cartesian and radial coordinates is $x = r\,\cos\varphi$, $y = r\,\sin\varphi$. Starting from this relation, derive the expression for the two-dimensional Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in radial coordinates. Show that

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

6. For afficionados: Consider a particle of mass M in a two-dimensional, radially symmetric, infinitely deep well with potential

$$\begin{aligned} V(r) &= 0 \qquad \text{for} \quad r < r_d\,, \\ V(r) &= \infty \qquad \text{for} \quad r > r_d\,. \end{aligned} \tag{1}$$

i) Write down the Schrödinger equation for this system. Use radial coordinates. ii) Seek a solution of the Schrödinger equation with the factorized ansatz $\Psi(r,\varphi) = R(r)\,\Phi(\varphi)$. What are the separate differential equations for $R(r)$ and $\Phi(\varphi)$? iii) Give the solution for $\Phi(\varphi)$. iv) Show that the differential equation of $R(r)$ is a solution to Bessel's differential equation

$$z^2\frac{\partial^2}{\partial z^2}R(z) + z\frac{\partial}{\partial z}R(z) + \left(z^2 - m^2\right)\,R(z) = 0$$

What is the interpretation of m and which values can it take? Look up the solutions of this differential equation in the library (Abramowitz and Stegun), and sketch their properties. Depending on how curious you are (and how many plus points you want to get), try to calculate the few lowest energy levels.