Solutions Homework 6

1a) 1-dim time-indep. Schrödinger eq.

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) = E\Psi(x)$$

$$\frac{\partial^2}{\partial x^2}\left(A\cos kx + B\sin kx\right) = -k^2(A\cos kx + B\sin kx)$$

$$\Rightarrow E = \frac{\hbar^2 k^2}{2m}$$

1b) $\Psi(x)$ is normalized:

$$1 = \int_{-\infty}^{\infty} dx\,|\Psi(x)|^2 = A^2 \int_{-\frac{L}{4}}^{\frac{L}{4}} \cos^2\left(\frac{2\pi x}{L}\right)dx$$

$$= A^2 \frac{1}{2}\int_{-\frac{L}{4}}^{\frac{L}{4}} \left(\cos^2\left(\frac{2\pi x}{L}\right) + \sin^2\left(\frac{2\pi x}{L}\right)\right)dx$$

area under $\cos^2$ and $\sin^2$ is the same

$$= A^2 \cdot \frac{L}{4}$$

i) $\Rightarrow A = \left(\frac{1}{2}\sqrt{L}\right)^{-1} = \frac{2}{\sqrt{L}}$

ii) Probability of measuring particle between $x \in [0, \frac{L}{8}]$

$$\int_0^{\frac{L}{8}} |\Psi|^2 dx = \left(\frac{L}{4}\right)^{-1}\int_0^{\frac{L}{8}} \cos^2\left(\frac{2\pi x}{L}\right)dx \qquad x = Lx'$$

$$= 4\int_0^{\frac{1}{8}} \cos^2(2\pi x')\,dx' = \frac{2}{\pi}\int_0^{\frac{\pi}{4}} \cos^2(x')\,dx'$$

$$= \frac{2}{\pi}\left(\frac{x'}{2} + \frac{1}{4}\sin(2x')\right)\Big|_{x'=0}^{\frac{\pi}{4}} = \underline{\underline{0.409}}$$

2. Quantum levels of bead

use (4.26) of lecture notes, E_n for 1-dim square well

$$E_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2 = \frac{(4.136 \cdot 10^{-15} eV \cdot s)^2}{8 \cdot 5g \cdot (20 cm)^2} n^2$$

$$\overset{?}{=} \frac{1}{2} m v^2$$

$$n^2 = \frac{4 \cdot m^2 L^2 v^2}{h^2} = \frac{4 \cdot (5 \cdot 10^{-3} kg)^2 \cdot (0.2m)^2 \cdot \left(\frac{10^{-10}}{3.65 \cdot 24 \cdot 36 \cdot 10^6} \frac{m}{s}\right)^2}{(4.136 \cdot 10^{-15} eV \cdot s)^2}$$

$$\left[\text{use } 1 eV = 5.344 \cdot 10^{-28} kg \frac{m}{s} \cdot 3 \cdot 10^8 \frac{m}{s} = 1.603 \cdot 10^{-19} kg \frac{m^2}{s^2} \right.$$

$$= \frac{4 \cdot 10^{-6} kg^2 m^2 \cdot (3.17 \cdot 10^{-18})^2 \frac{m^2}{s^2}}{(4.136 \cdot 1.603 \cdot 10^{-34})^2 kg^2 \frac{m^4}{s^2}} = (0.478)^2 \left(\frac{10^{-21}}{10^{-34}}\right)^2 \cdot 4$$

$$n = 0.478 \cdot 10^{13} \cdot 4 = 1.91 \cdot 10^{-13}$$

3. $$\Delta E = E_2 - E_1 = \frac{\hbar^2 \pi^2}{2 m_p L^2} (2^2 - 1^2) = h \cdot \frac{c}{\lambda}$$

$\frac{h^2}{8 m_p L^2}$

$$\lambda = \frac{8 m_p \cdot L^2 \cdot c}{3 \ h} = \frac{4 \cdot 2 \cdot 938.3 MeV/c^2 \cdot (10^{-14} m)^2 \cdot c}{3 \cdot 4.136 \cdot 10^{-15} eV \cdot s}$$

$$= \frac{4 \cdot 2 \cdot 9.383 \cdot 10^8 eV \cdot (10^{-14} m)^2}{3 \cdot 4.136 \cdot 10^{-15} eV s \cdot 3 \cdot 10^8 \frac{m}{s}} = \frac{4 \cdot 2}{9} \cdot \frac{9.383}{4.136} 10^{-13} m$$

$$= \cancel{1.51 \cdot 10^{-13} m} \ 4 \cdot 0.5 \cdot 10^{-13} m = 2 \cdot 10^{-13} m$$

4. $$E_n = \frac{\hbar^2 n^2 \pi^2}{2 m_e L^2} = \frac{h^2}{8 m_e L^2} n^2$$

$$E_1 = \frac{\left(4.136 \cdot 10^{-15} eV \cdot s\right)^2}{8 \cdot 0.511 \cdot 10^{6} eV \cdot (10^{-10} m)^2} \cdot \cancel{c} \cdot \left(3 \cdot 10^8 \tfrac{m}{s}\right)^2$$

$$= \frac{\left(4.136 \cdot 10^{-5}\right)^2 \cdot \left(3 \cdot 10^8\right)^2}{\cancel{8}\, 2 \cdot (2 \cdot 10^3)^2 \cdot 0.511} \; eV$$

$$= \frac{1}{2 \cdot 0.511} \cdot \left(\frac{4.136 \cdot 3}{2}\right)^2 eV = 37.66 \, eV$$

Energy levels: $\Delta E = h \frac{c}{\lambda} = E_{n_i} - E_{n_f} = 37.66 eV \cdot (n_i^2 - n_f^2)$

$$\lambda_{n_i n_f} = \frac{hc}{37.66 eV} (n_i^2 - n_f^2)$$

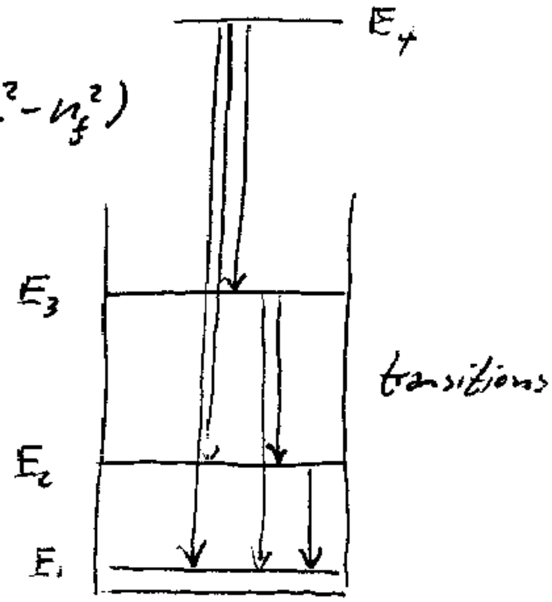
$$= \frac{4.136 \cdot 10^{-15} eV \cdot 3 \cdot 10^8 \frac{m}{s}}{37.66 eV} (n_i^2 - n_f^2)$$

$$= \left(0.329 \cdot 10^{-7} m\right) (n_i^2 - n_f^2)$$

choices: $n_i = 4 \quad n_f = 3, 2, 1$

$n_i = 3 \quad n_f = 2, 1$

$n_i = 2 \quad n_f = 1$



5.) again $E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2} = \frac{h^2}{8 m_e L^2} n^2$

$\Delta E = \frac{h^2}{8 m_e L^2}(2^2 - 1^2) = h\frac{c}{\lambda}$

$\cancel{\lambda = 8 m_e L^2}$ $L^2 = \frac{3h^2}{8 m_e} \cdot \frac{\lambda}{hc}$

$L^2 = \frac{3}{8} \cdot \frac{h\lambda}{m_e c} = \frac{3}{8} \cdot \frac{4.136 \cdot 10^{-15} eVs \cdot 694.3 \cdot 10^{-9} m}{0.511 \cdot 10^6 eV} \cdot 3 \cdot 10^8 \frac{m}{s}$

$= \frac{3}{8} \cdot \frac{4.136 \cdot 6.943}{0.511} \cdot 10^{-15-7+8-6} \; m^2$

$L = \sqrt{\frac{3}{8} \, \frac{4.136 \cdot 6.943}{0.511}} \; 10^{-10} m = 4.59 \cdot 10^{-10} m$

6.) Eigenfunctions in infinite square well are

$\phi_n(x) = A_n \sin k_n x$, $k_n = \frac{n\pi}{L}$, $0 < x < L$

Norm: $1 = \int_0^L dx \, |\phi(x)|^2 = A_n^2 \int_0^L dx \sin^2\left(\frac{n\pi x}{L}\right) = A_n^2 \frac{L}{2}$

$\Rightarrow A_n = \sqrt{2} \frac{1}{\sqrt{L}}$

$\langle x \rangle = 2 \int_0^L dx \; x \sin^2\left(\frac{n\pi x}{L}\right) \frac{1}{L} = 2L \underbrace{\int_0^1 dx \; x \sin^2(n\pi x)}_{1/4}$

$= \frac{L}{2}$

$$\langle x^2 \rangle = A_n^2 \int_0^L dx \; x^2 \sin^2\left(\frac{n\pi x}{L}\right)$$

$$= \frac{2}{L} L^3 \int_0^1 dx' \; x'^2 \sin^2(n\pi x')$$

$$= 2L^2 \left(\frac{1}{6} - \frac{1}{4n^2\pi^2}\right)$$

$$= \frac{L^2}{3} - \frac{L^2}{2(n\pi)^2}$$

For curiosity: $\langle x^2 \rangle - \langle x \rangle^2 = \frac{L^2}{3} - \frac{L^2}{2(n\pi)^2} - \frac{L^2}{4}$

~~= yes~~ This defines the 'uncertainty'

$$\Delta x = \sqrt{|\langle x^2 \rangle - \langle x \rangle^2|}$$